Scientific Voyage
Vol. 1, No. 4,
Page 10-23 (2020)

SCIENTIFIC VOYAGE
ISSN: 2395-5546

GCECT
Publication
Oct.-Dec.

Hawking radiation with the dynamical horizon in the K-essence emergent Vaidya spacetime

Bivash Majumder[a1], Goutam Manna[b2] and Mrinal Manna[c3]

[a] *Department of Mathematics, Prabhat Kumar College, Contai, Purba Medinipur 721404, West Bengal, India*
[b] *Department of Physics, Prabhat Kumar College, Contai, Purba Medinipur 721404, West Bengal, India*
[c] *Department of Physics, Ramakrishna Mission Vivekananda Educational and Research Institute, Belur Math, Howrah 711202, West Bengal, India*

Abstract: We study the Hawking radiation with the dynamical horizon in the **K**-essence Vaidya geometry. By considering the **K**-essence action to be of the Dirac-Born-Infeld variety, the physical spacetime to be a general static spherically symmetric black hole, and by restricting the **K**-essence scalar field to be a function solely of the advanced or the retarded time, Manna et. al. have established the connection between the **K**-essence emergent gravity scenario and generalizations of Vaidya spacetime. Based on modified definition of the dynamical horizon by Sawayama, we investigate the Hawking effect in the **K**-essence Vaidya Schwarzschild spacetime. Especially, we investigate the Hawking Radiation in the two ways, by using the dynamical horizon equation and using the tunneling formalism. The results are different from the usual Vaidya spacetime geometry.

Keywords: Emergent gravity; **K**-essence; Vaidya spacetimes

1. Introduction

It is believed that a black hole is formed by the collapse of matter and it radiates the thermal radiation whose temperature is proportional to the surface gravity [1–10]. In [1], it is assumed that the spacetime is to be static or stationary for calculating Hawking Radiation. This assumption will be valid only when the radiated energy is so small compared to the mass energy of the black hole. When the radiation becomes sufficiently large, it can be modified via Einstein equation. In this context, Vaidya [11,12] has solved nonstatic solution of the Einstein's field equations for spheres of fluids radiating energy. The nonstatic analogs of Schwarzschild's interior solution in General Relativity (GR) has been established in [13,14] and the problem of gravitational collapse with radiation has been solved in [15]. The solution has satisfied the physical feature of allowing a positive definite value of the density of collapsing matter, and it gives the total luminosity of the object as observed by a stationary observer at infinity to be zero when the collapsing object approaches to the Schwarzschilds singularity. So, we can say that the Vaidya spacetime [11–15] is a non-stationary Schwarzschild spacetime. Husain [16] and Wang et. al. [17] have developed the generalizations of Vaidya spacetime corresponding

1 E-mail: bivashmajumder@gmail.com
2 E-mail: goutammanna.pkc@gmail.com
3 E-mail: mrinalmanna1997@gmail.com

to the gravitational collapse of a null fluid. Recently, Manna et al. [18,19] have established the **K**-essence generalizations of Vaidya spacetime where time dependence of the metric comes from the kinetic energy (ϕ_v^2) of the **K**-essence scalar field (ϕ). The Hawking radiation [1–10] has been discussed in [20–24] using tunneling mechanism. Also, it was developed by the method of complex path analysis which is used to describe tunnelling processes in semiclassical quantum mechanics in [25,26]. Kerner and Mann [27] have established, in general, that the Hawking temperature is independent of the angular part of the spacetime. In [28–31], they have discussed the Hawking radiation of Vaidya or Vaidya-Bonner spacetime.

The **K**-essence theory [32–38] is a scalar field theory where the kinetic energy of the field dominates over the potential energy of the field. The differences between the relativistic field theories with canonical kinetic terms and the **K**-essence theory with non-canonical kinetic terms are that the non-trivial dynamical solutions of the K-essence equation of motion, which not only spontaneously break Lorentz invariance but also change the metric for the perturbations around these solutions. Thus the perturbations propagate in the so called *emergent or analogue* curved spacetime with the metric. Based on the Dirac-Born-Infeld (DBI) model [39–41], Manna et al. [42–45] have developed the simplest form of **K**-essence emergent gravity metric $\bar{G}_{\mu\nu}$ which is not conformally equivalent to the usual gravitational metric $g_{\mu\nu}$. The theoretical form in the **K**-essence field, the lagrangian is non-canonical and it does not depend explicitly on the field itself. The general form of the Lagrangian for the **K**-essence model is: $L = -V(\phi)F(X)$ where $X = \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$.

Ashtekar and Krishnan have considered the dynamical horizons in their paper [46–51], and derived a new equation that describe how the dynamical horizon radius changes. The definition of dynamical horizon is as follows: *Dynamical Horizon*: A smooth, three-dimensional, space-like submanifold H in a space-time $\mathcal{M}$ is said to be a dynamical horizon if it can be foliated by a family of closed $2-$surfaces such that, on each leaf S, the expansion $\Theta_{(l)}$ of one null normal l^a vanishes and the expansion $\Theta_{(n)}$ of the other null normal n^a is strictly negative. Also, *this definition can be modified as [52]:* A smooth, three-dimensional, spacelike or timelike submanifold H in a space-time is said to be a dynamical horizon if it is foliated by a preferred family of 2-spheres such that, on each leaf S, the expansion $\Theta_{(l)}$ of a null normal l^a vanishes and the expansion $\Theta_{(n)}$ of the other null normal n^a is strictly negative.

Following [48], in the concept of world tubes, if the marginally trapped tubes (MTT) is spacelike,it is called a dynamical horizon (DH). Under some conditions, it provides a quasi-local representation of an evolving black hole. If it is null, it describes a quasi-local description of a black hole in equilibrium and is called an isolated horizon (IH). If the MTT is timelike, causal curves can transverse it in both inward and outward directions, where it does not represent the surface of a black hole in any useful sense, it is called a timelike membrane (TLM).

In this work, we have studied the Hawking radiation with the dynamical horizon in the **K**-essence emergent Vaidya spacetime based on Sawayama [52] by considering dynamical horizon equation [46–49] and tunneling formalism [20–26,28, 29,42–44].

The paper is organized as follows: In section 2, we briefly discuss the **K**-essence emergent geometry and corresponding **K**-essence Vaidya spacetime. We describe the dynamical horizons considering Schwarzschild black hole as a background for the **K**-essence emergent Vaidya spacetime in section 3. In the next section, we have discussed in detail the dynamical horizon equation for the **K**-essence Vaidya Schwarzschild spacetime. Also, we have discussed the corresponding Hawking radiation using dynamical horizon equation and tunneling mechanism. The Last section is our discussion.

2. Brief review of K-essence and K-essence-Vaidya Geometry

The **K**-essence scalar field ϕ minimally coupled to the background gravitational metric $g_{\mu\nu}$ has action [32]-[36]

$$S_k[\phi, g_{\mu\nu}] = \int d^4x\sqrt{-g}L(X,\phi), \tag{1}$$

where $X = \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ and the energy-momentum tensor is

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_k}{\delta g^{\mu\nu}} = L_X\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}L, \tag{2}$$

where $L_{\mathrm{X}} = \frac{dL}{dX}$, $L_{XX} = \frac{d^2L}{dX^2}$, $L_\phi = \frac{dL}{d\phi}$ and ∇_μ is the covariant derivative defined with respect to the gravitational metric $g_{\mu\nu}$.

The scalar field equation of motion is

$$-\frac{1}{\sqrt{-g}}\frac{\delta S_k}{\delta\phi} = \tilde{G}^{\mu\nu}\nabla_\mu\nabla_\nu\phi + 2XL_{X\phi} - L_\phi = 0, \tag{3}$$

where

$$\tilde{G}^{\mu\nu} \equiv L_X g^{\mu\nu} + L_{XX}\nabla^\mu\phi\nabla^\nu\phi, \tag{4}$$

and $1 + \frac{2XL_{XX}}{L_X} > 0$.

Using the conformal transformations $G^{\mu\nu} \equiv \frac{c_s}{L_x^2}\tilde{G}^{\mu\nu}$ and $\bar{G}_{\mu\nu} \equiv \frac{c_s}{L_X}G_{\mu\nu}$, with $c_s^2(X,\phi) \equiv (1 + 2X\frac{L_{XX}}{L_X})^{-1}$ we have [42–44]

$$\bar{G}_{\mu\nu} = g_{\mu\nu} - \frac{L_{XX}}{L_X + 2XL_{XX}}\nabla_\mu\phi\nabla_\nu\phi. \tag{5}$$

Here one must always have $L_X \neq 0$ for c_s^2 to be positive definite and only then equations $(1) - (4)$ will be physically meaningful.

If L is not an explicit function of ϕ then the equation of motion (3) reduces to

$$-\frac{1}{\sqrt{-g}}\frac{\delta S_k}{\delta\phi} = \bar{G}^{\mu\nu}\nabla_\mu\nabla_\nu\phi = 0. \tag{6}$$

Note that for non-trivial spacetime configurations of ϕ, the emergent metric $G_{\mu\nu}$ is, in general, not conformally equivalent to $g_{\mu\nu}$. So ϕ has properties different from canonical scalar fields, with the local causal structure also different from those defined with $g_{\mu\nu}$. We consider the DBI type Lagrangian as [42–44, 39–41]

$$L(X,\phi) = 1 - V(\phi)\sqrt{1-2X}, \tag{7}$$

for $V(\phi) = V$ =constant and kinetic energy of $\phi >> V$, i.e. $(\dot{\phi})^2 >> V$. This is a typical for the **K**-essence fields where the kinetic energy dominates over the potential energy. Then $c_s^2(X,\phi) = 1 - 2X$. For scalar fields $\nabla_\mu \phi = \partial_\mu \phi$. Thus (5) becomes

$$\bar{G}_{\mu\nu} = g_{\mu\nu} - \partial_\mu \phi \partial_\nu \phi. \tag{8}$$

The corresponding geodesic equation for the **K**-essence theory in terms of the new Christoffel connections $\bar{\Gamma}$ is [42–44]

$$\frac{d^2x^\alpha}{d\lambda^2} + \bar{\Gamma}^\alpha_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0, \tag{9}$$

where λ is an affine parameter and

$$\begin{aligned}\bar{\Gamma}^\alpha_{\mu\nu} &= \Gamma^\alpha_{\mu\nu} + \Big(1-2X\Big)^{-1/2}\bar{G}^{\alpha\gamma}\Big[\bar{G}_{\mu\gamma}\partial_\nu\Big(1-2X\Big)^{1/2} \\ &+ \bar{G}_{\nu\gamma}\partial_\mu\Big(1-2X\Big)^{1/2} - \bar{G}_{\mu\nu}\partial_\gamma\Big(1-2X\Big)^{1/2}\Big] \\ &= \Gamma^\alpha_{\mu\nu} - \frac{1}{2(1-2X)}\Big[\delta^\alpha_\mu\partial_\nu X + \delta^\alpha_\mu\partial_\nu X\Big].\end{aligned} \tag{10}$$

2.1 **K***-essence-Vaidya Geometry*

Considering a general spherically symmetric static (black hole) Eddington-Finkelstein line element [18]

$$ds^2 = f(r)dv^2 - 2\epsilon dvdr - r^2 d\Omega^2, \tag{11}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\Phi^2$. When $\epsilon = +1$, the null coordinate v represents the Eddington advanced time (outgoing), while when $\epsilon = -1$, it represents the Eddington retarded time (incoming).

From (8) the emergent spacetime is described by the line element

$$dS^2 = ds^2 - \partial_\mu\phi\partial_\nu\phi dx^\mu dx^\nu. \tag{12}$$

Considering the scalar field $\phi(x) = \phi(v)$, so that the emergent spacetime line element is

$$dS^2 = \Big[f(r) - \phi_v^2\Big]dv^2 - 2\epsilon dvdr - r^2 d\Omega^2, \tag{13}$$

where $\phi_v = \frac{\partial\phi}{\partial v}$.

Notice that this assumption on ϕ actually violates local Lorentz invariance, since in general, spherical symmetry would only require that $\phi(x) = \phi(v, r)$. But in the **K**-essence theory, the dynamical solutions are spontaneously break Lorentz invariance. So, in this context, our choice of form of the **K**-essence scalar field is physically permissible.

Now comparing the emergent spacetime (13) with the metric [16,17] of the generalized Vaidya spacetimes corresponding to gravitational collapse of a null fluid (take $\epsilon = +1$)

$$dS_V^2 = \left(1 - \frac{2m(v,r)}{r}\right)dv^2 - 2dvdr - r^2 d\Omega^2, \tag{14}$$

where the mass function is

$$m(v.r) = \frac{1}{2} r \Big[1 + \phi_v^2 - f(r) \Big]. \tag{15}$$

These forms (13) or (14) of metrics are satisfying all the required energy conditions [53] (weak, strong, dominant) for generalizations of the **K**-essence emergent spacetime with the Vaidya spacetime provided $\phi_v \phi_{vv} > 0$; $1+\phi_v^2 > f + r f_r$; $2f_r + r f_{rr} > 0$ which have established by Manna et al. [18].

3. Dynamical horizons

Following Ashtekar and Krishnan [47] and Sawayama [52], we can discuss behavior of the dynamical horizon of the **K**-essence emergent Vaidya spacetime. The generalized Vaidya spacetime (13) or (14) in the presence of kinetic energy of the **K**-essence scalar field ϕ_v^2 can be written as

$$dS^2 = F(v,r) dv^2 - 2dvdr - r^2 d\Omega^2, \tag{16}$$

where v^a is a null vector.

Following Sawayama [52], we can define

$$a = \frac{dr}{dr^*}, \tag{17}$$

where r^* is tortoise coordinate defined as $v = t + r^*$.

There are two null vectors,

$$l^a = \begin{bmatrix} l^t \\ l^{r^*} \\ l^\theta \\ l^\Phi \end{bmatrix} = \begin{bmatrix} a^{-1} \\ -a^{-1} \\ 0 \\ 0 \end{bmatrix}, \tag{18}$$

corresponding to the null vector v^a , and the other is

$$n^a = \begin{bmatrix} n^t \\ n^{r^*} \\ n^\theta \\ n^\Phi \end{bmatrix} = \begin{bmatrix} a^{-1} \\ \frac{F}{F-2a} a^{-1} \\ 0 \\ 0 \end{bmatrix}. \tag{19}$$

The expansions $\Theta_{(l)}$ and $\Theta_{(n)}$ of the two null vectors l^a and n^a are [52]

$$\Theta_{(l)} = \frac{1}{r}(2F - a), \tag{20}$$

and

$$\Theta_{(n)} = \frac{1}{ar} \left(\frac{-2F^2 + aF - 2a^2}{-F + 2a} \right). \tag{21}$$

We can see that as $\Theta_{(l)} = 0 \Rightarrow 2F - a = 0$, and the other null expansion $\Theta_{(n)}$ is strictly negative which are the required conditions for the dynamical horizon [46, 47, 52]. So the horizons of our case are dynamical. Note that, based on Ashtekar et al. [46–50], Manna et al. [18] also have established the possibility of the dynamical horizon for the **K**-essence emergent Vaidya spacetime.

3.1 Schwarzschild black hole as background

In this case $f(r) = 1 - \frac{2M}{r}$, then (16) become

$$dS^2 = \Big(1 - \frac{2M}{r} - \phi_v^2\Big)dv^2 - 2dvdr - r^2 d\Omega^2, \tag{22}$$

with

$$F = \Big(1 - \frac{2M}{r} - \phi_v^2\Big) \equiv \Big(1 - \frac{2m(v,r)}{r}\Big), \tag{23}$$

and the mass function become

$$m(v,r) = M + \frac{r}{2}\phi_v^2, \tag{24}$$

where the tortoise coordinate

$$r^* = \frac{1}{N}\Big[r + B\ ln(r - B)\Big], \tag{25}$$

where $N \equiv N(v) = (1 - \phi_v^2)$, $B = 2M'(v) = \frac{2M}{N}$ with $M'(v) = \frac{M}{1-\phi_v^2}$.

Note that in the above spacetime (22) always $\phi_v^2 < 1$. If $\phi_v^2 > 1$, the signature of this spacetime will be ill-defined. Also the condition $\phi_v^2 \neq 0$ holds good instead of $\phi_v^2 = 0$, which leads to non-applicability of the **K**-essence theory.

Now differentiating the above equation (25) with respect to r^* we get

$$\begin{aligned} a = F\Big[1 - \frac{1}{N}\Big(\frac{dB}{dv}\ ln(r - B) - \frac{B}{r - B}\frac{dB}{dv}\Big) \\ + \frac{1}{N^2}\Big(r + B\ ln(r - B)\frac{dN}{dv}\Big)\Big]. \end{aligned} \tag{26}$$

Now we solve $\Theta_{(l)} = 0$, to determine the dynamical horizon radius as

$$\begin{aligned} 2F - a = 2F - F\Big[1 - \frac{1}{N}\Big(\frac{dB}{dv}\ ln(r - B) - \frac{B}{r - B}\frac{dB}{dv}\Big) \\ + \frac{1}{N^2}\Big(r + B\ ln(r - B)\frac{dN}{dv}\Big)\Big] = 0. \end{aligned} \tag{27}$$

From the above equation we have two solutions at $r = r_D$ one is $F = 0$, i.e.,

$$r_D = 2M'(v) = \frac{2M}{1 - \phi_v^2}, \tag{28}$$

and another is

$$\begin{aligned} 1 + \frac{d}{dv}\Big[\frac{B}{N}\ ln(r_D - B)\Big] - \frac{r_D}{N^2}\frac{dN}{dv} = 0 \\ \Rightarrow (r_D - B)^B e^{r_D} = e^{-vN}. \end{aligned} \tag{29}$$

The value of r_D can be written in terms of the Wright ω function [55] as

$$r_D = B\Big[1 + \omega(Z)\Big] = \frac{2M}{N}\Big[1 + \omega(Z)\Big], \tag{30}$$

with $Z = -[1 + ln(B) + vC]$, $C = \frac{N^2}{2M} = \frac{(1-\phi_v^2)^2}{2M}$.

The Wright ω function is a single-valued function, defined in terms of the multi-valued Lambert W function [56] as $\omega(Z) = W_{\mathcal{K}(Z)}(e^Z)$ where $\mathcal{K}(Z)(= [\frac{(Im(Z)-\pi)}{2\pi}])$

is the unwinding number of Z. The sign of this unwinding number is such that $ln(e^Z) = Z + 2\pi i\mathcal{K}(Z)$ which is opposite to the sign used in [57]. The algebraic properties [55] of the Wright ω function are

$$\frac{d\omega}{dZ} = \frac{\omega}{1+\omega}, \tag{31}$$

$$\int \omega^n dZ = \begin{cases} \dfrac{\omega^{n+1}-1}{n+1} + \dfrac{\omega^n}{n} & if\ n \neq -1 \\ ln\ \omega - \dfrac{1}{\omega} & if\ n = -1, \end{cases} \tag{32}$$

with the analytic property $Z = \omega + ln\ \omega$.

Here mention that if we consider the usual Vaidya spacetime without **K**-essence scalar field ϕ i.e., $m(v,r) \equiv M(v)$ and $\phi_v^2 = 0$, then from the above equation (29) we can get back to the Sawayama [52] result as

$$r_D = 2M(v) + e^{-v/2M(v)}. \tag{33}$$

4. Dynamical horizon equation and Hawking radiation

From (16), we can construct the Ricci scalar ($\bar{R}$) and the Ricci tensors ($\bar{R}_{\mu\nu}$) of the **K**-essence emergent Vaidya spacetime as

$$\begin{aligned} \bar{R}_{vv} &= \frac{1}{r}\partial_v F - \frac{F}{2}\partial_r^2 F - \frac{F}{r}\partial_r F\ : \\ \bar{R}_{rv} = \bar{R}_{vr} &= \frac{1}{2}\partial_r^2 F + \frac{1}{r}\partial_r F\ :\ \bar{R}_{rr} = 0\ ; \\ \bar{R}_{\theta\theta} &= F + r\partial_r F - 1\ ;\ \bar{R}_{\Phi\Phi} = sin^2\theta \bar{R}_{\theta\theta}\ ; \\ \bar{R} &= -[\partial_r^2 F + \frac{4}{r}\partial_r F + \frac{2}{r^2}(F-1)]. \end{aligned} \tag{34}$$

Using these values of the Ricci scalar and the Ricci tensors (34), we can construct the components of the energy momentum tensor for the **K**-essence emergent Vaidya spacetime through the "emergent" Einstein's equation $\bar{R}_{\mu\nu} - \frac{1}{2}\bar{G}_{\mu\nu}\bar{R} = 8\pi\bar{T}_{\mu\nu}$, taking the gravitational constant $G = 1$, as

$$8\pi\bar{T}_{vv} = \frac{1}{r}\partial_v F + \frac{1}{r}F\partial_r F + \frac{F}{r^2}(F-1), \tag{35}$$

$$8\pi\bar{T}_{vr} = -[\frac{1}{r}\partial_r F + \frac{1}{r^2}(F-1)], \tag{36}$$

$$8\pi\bar{T}_{rr} = 0, \tag{37}$$

$$8\pi\bar{T}_{\theta\theta} = -[\frac{r^2}{2}\partial_r^2 F + r\partial_r F]\ ;\ 8\pi\bar{T}_{\Phi\Phi} = sin^2\theta\bar{T}_{\theta\theta}. \tag{38}$$

Now, *for the Schwarzschild background case (i.e. subcase 3.1)*, the components of the energy-momentum tensors for the spherically symmetric **K**-essence emergent

Schwarzschild Vaidya spacetime (22) are

$$\begin{aligned}
8\pi\bar{T}_{vv} &= -\frac{2}{r^2}[\partial_v m + F\partial_r m] \\
&= -\left[\frac{2\phi_v\phi_{vv}}{r} + \frac{\phi_v^2}{r^2}(1 - \frac{2M}{r} - \phi_v^2)\right], \qquad (39)\\
8\pi\bar{T}_{vr} &= \frac{2}{r^2}\partial_r m = \frac{\phi_v^2}{r^2} \\
or\ 8\pi\bar{T}_{vr^*} &= \frac{2a}{r^2}\partial_r m = a\frac{\phi_v^2}{r^2}, \qquad (40)\\
8\pi\bar{T}_{rr} &= 8\pi\bar{T}_{r^*r^*} = 0. \qquad (41)
\end{aligned}$$

Following Sawayama [52], we can derive the energy-momentum tensor $\bar{T}_{\hat{t}l}$ for the integration of the dynamical horizon equation exactly stated in [46,47,52] as

$$\begin{aligned}
\frac{1}{2G}(R_2 - R_1) &= \int_{\Delta H} T_{ab}\hat{\tau}^a\xi^b_{(R)}d^3V \\
&+\frac{1}{16\pi G}\int_{\Delta H} N_R[|\sigma|^2 + 2|\zeta|^2]d^3V, \qquad (42)
\end{aligned}$$

where R_2, R_1 are the radii of the dynamical horizon, T_{ab} is the stress-energy tensor, $|\sigma|^2 = \sigma_{ab}\sigma^{ab}$, $|\zeta|^2 = \zeta_a\zeta^a$, σ_{ab} is the shear, $\zeta^a = \tilde{q}^{ab}\hat{r}^c\nabla_c l_b$, with the two-dimensional metric $\tilde{q}^{ab}$, and $\xi^a_{(R)} = N_R l^a$ with $N_R = |\partial R|$, where R is the radius of the dynamical horizon. This is the dynamical horizon equation that tells us how the horizon radius changes due to the matter flow, shear, and expansion. In the spherically symmetric case, the second term of the right-hand side of the dynamical equation vanishes.

At first, we can write $\bar{T}_{tl}$ in terms of $\bar{T}_{vv}$ and $\bar{T}_{vr^*}$ as

$$\begin{aligned}
\bar{T}_{tl} = \bar{T}_{vv} - \bar{T}_{vr^*} &= -\frac{1}{4\pi r^2}\frac{5}{2}\partial_v m \\
&= -(\frac{1}{4\pi r^2})\frac{5}{2}r\phi_v\phi_{vv}. \qquad (43)
\end{aligned}$$

Considering $\hat{t}^a$ being the unit vector in the direction of t^a, then we have

$$\bar{T}_{\hat{t}l} = -\frac{1}{4\pi r^2}\frac{5}{2}(\partial_v m)F^{-1}, \qquad (44)$$

with F is defined in (23). At the horizon, the expression of $\bar{T}_{\hat{t}l}$ is $\bar{T}_{\hat{t}l}\Big|_{r=r_D} = \frac{1}{4\pi r_D^2}\frac{5r_D}{4}(\partial_v N)F^{-1}$ where we have used $m \equiv m(v,r) = M + \frac{r}{2}(1-N)$.

To evaluate, the dynamical horizon integration (42) in terms of the Wright ω function, at first, we derive some terms as follows:

Using equation (30), we get

$$\begin{aligned}
\frac{dr_D}{dv} = \Bigg[&-\frac{2M}{N^2}\Big(1+\omega(Z)\Big) + \frac{2M}{N}\,\frac{\omega(Z)}{1+\omega(Z)} \times \\
&\Big(\frac{1}{N} - \frac{Nv}{M}\Big)\Bigg]\partial_v N - \frac{N\,\omega(Z)}{1+\omega(Z)}. \qquad (45)
\end{aligned}$$

At the horizon, we obtain, from the above equation (45)

$$\partial_v N = \frac{N\,\omega(Z)}{1+\omega(Z)}\Bigg[-\frac{2M}{N^2}\Big(1+\omega(Z)\Big) + \frac{2M}{N}\frac{\omega(Z)}{1+\omega(Z)}\Big(\frac{1}{N}-\frac{Nv}{M}\Big)\Bigg]^{-1}. \quad (46)$$

and from (30)

$$\frac{dr_D}{dN} = \Big(\frac{N}{\partial_v N}\Big)\frac{\omega(Z)}{1+\omega(Z)}. \quad (47)$$

Now, re-write the equation (23) in terms of Write ω function as

$$F = \frac{N\,\omega(Z)}{1+\omega(Z)}, \quad (48)$$

and from the equation (24)

$$\partial_v m = -\Big(\frac{r_D}{2}\Big)\partial_v N. \quad (49)$$

Changing the order of integration r_D to N in dynamical horizon integration (42), we have

$$\frac{1}{2}\left[\frac{2M}{N}(1+\omega(Z))\right]\Bigg|_{N_1}^{N_2} = \frac{5}{4}\int_{N_1}^{N_2}\frac{2M}{N}(1+\omega(Z))dN, \quad (50)$$

since in the above calculation, the functions $\partial_v N$ and F^{-1} with fixed r_D are used only in the integration.

Now, using the equations (44) to (49) and inserting the above equation (50) in the equation (42) and taking the limit $N_2 \to N_1 = N$, we have

$$-\frac{M}{N^2}\Big(1+\omega(Z)\Big) + \frac{M}{N}\frac{\omega(Z)}{(1+\omega(Z))}\Big(\frac{1}{N}-\frac{Nv}{M}\Big) - \frac{5}{2}\frac{M}{N}\Big(1+\omega(Z)\Big) = 0, \quad (51)$$

which is the dynamical horizon equation for the spherically symmetric **K**-essence emergent Schwarzschild Vaidya spacetime. This dynamical horizon equation in the presence of the kinetic energy of the **K**-essence scalar field is far different from the usual Vaidya dynamical horizon equation [52].

Now *we discuss about the Hawking radiation [1–10]:*
To solve this problem, we consider two ideas, which are (1) to use the dynamical horizon equation (42), and (2) to use the **K**-essence Schwarzschild Vaidya metric (22) using tunneling mechanism [20–26, 28, 29].

4.1 Using the dynamical horizon equation
Considering the result of Candelas [54], for the matter on the dynamical horizon,

which assumes that spacetime is almost static and is valid near the horizon [52], from (22), $r \sim 2m(\equiv \frac{2M}{N})$

$$\bar{T}_{tl} = \frac{-1}{2\pi^2(1-\frac{2m}{r})}\int_0^\infty \frac{w^3 dw}{e^{8\pi m w}-1} = \frac{-1}{2m^4\pi^2 c(1-\frac{2m}{r})}, \tag{52}$$

with $c = 15\times 8^4 = 61440$, where we have used the following value of the integration $\int_0^\infty \frac{w^3 dw}{e^{bw}-1} = \frac{\pi^4}{15b^4}$.

The matter energy of the equation (52) is negative near $r \sim 2m$, which means that in the dynamical horizon equation, the **K**-essence Schwarzschild Vaidya black hole absorbs negative energy, i.e., the black hole radius decreases. Note that, the **K**-essence Schwarzschild Vaidya mass $m(\equiv \frac{M}{1-\phi_v^2})$, at near the horizon, is greater than the Schwarzschild mass as $\phi_v^2 < 1$. Also, the dynamic behavior of the mass function $m(v,r)$ is carried by the kinetic energy (ϕ_v^2) of the **K**-essence scalar field which is defined by the equation (24). Here we use the dynamical horizon equation, since we need only information about the matter near the horizon, without solving the full Einstein equation with the backreaction.

Now following [52], we can get

$$\bar{T}_{\hat{t}l} = \frac{1}{2m^4\pi^2 c(1-\frac{2m}{r})}. \tag{53}$$

Again, by changing the order of integration of the right hand side from r_D to N of the dynamical horizon equation (42)

$$\int_{r_1}^{r_2} 4\pi r_D^2 \bar{T}_{\hat{t}l} dr_D = b\int_{N_1}^{N_2}\Big[\frac{2M}{N}(1+\omega(Z))\Big]^2 \frac{(1+\omega(Z))}{(N\omega(Z))}\times \Big[M+\frac{M}{N}(1+\omega(Z))(1-N)\Big]^{-4}\Big(\frac{dr_D}{dN}\Big)dN$$
$$= b\int_{N_1}^{N_2}\Big[\frac{2M}{N}(1+\omega(Z))\Big]^2 \times \Big[M+\frac{M}{N}(1+\omega(Z))(1-N)\Big]^{-4}\Big(\frac{1}{\partial_v N}\Big)dN, \tag{54}$$

where $b = \frac{2}{\pi c}$ is a constant. Since in our case, the time dependence of the **K**-essence Schwarzschild Vaidya metric is carrying ϕ_v^2 i.e., $N(=1-\phi_v^2)$.

Now insert this integration (54) into the dynamical horizon equation (42), we obtain

$$\frac{1}{2}\Big[\frac{2M}{N}\Big(1+\omega(Z)\Big)\Big]\Big|_{N_1}^{N_2} = b\int_{N_1}^{N_2}\Big[\frac{2M}{N}\Big(1+\omega(Z)\Big)\Big]^2\times \Big[M+\frac{M}{N}\Big(1+\omega(Z)\Big)(1-N)\Big]^{-4}\Big(\frac{1}{\partial_v N}\Big)dN + \frac{5}{4}\int_{N_1}^{N_2}\frac{2M}{N}\Big(1+\omega(Z)\Big)dN. \tag{55}$$

Now taking the limit $N_2 \to N_1 = N$ and $\omega(Z) \neq 0$, the dynamical horizon equation becomes

$$M^2\left[1+\left(1-\frac{1}{N}\right)\left(1+\omega(Z)\right)\right]^4 = \frac{4b(1+\omega(Z))^3}{N^3\omega(Z)}\left[\frac{1}{2}-\frac{5(1+\omega(Z))}{4}\times\right.$$
$$\left.\left(-\frac{1+\omega(Z)}{N}+\frac{\omega(Z)}{1+\omega(Z)}\left(\frac{1}{N}-\frac{Nv}{M}\right)\right)^{-1}\right]^{-1}. \tag{56}$$

Here we are using the fact that $N \neq 0$ and $M \neq 0$. In this section, our main objective is to find the behavior of the mass of the black hole $m(v,r)$ with v for fixed M as in [52]. To conclude this, we have calculated the equation (56) to find $N(=1-\phi_v^2)$ as a function of z but this equation (56) is a transcendental equation and it cannot solve analytically or numerically since it includes the wright omega function and higher degrees of N. So that this equation (56) fails to find the behavior of the mass of the black hole $m(v,r)$ (for fixed M) through the relation $m(v,r) = M + \frac{r}{2}(1-N)$ at the horizon.

4.2 Using the tunneling formalism

To calculate the Hawking temperature using *tunnelling formalism* [20–26, 28, 29, 42–44], consider a massless particle in a black hole (22) background is described by the Klein-Gordon equation

$$\hbar^2\left(-\bar{G}\right)^{-1/2}\partial_\mu\left(\bar{G}^{\mu\nu}\left(-\bar{G}\right)^{1/2}\partial\nu\Psi\right), \tag{57}$$

where Ψ can be taken in the form

$$\Psi = exp\left(\frac{i}{\hbar}S+...\right), \tag{58}$$

to obtain the leading order in $\hbar$ the Hamilton-Jacobi equation is

$$\bar{G}^{\mu\nu}\partial_\mu S\partial_\nu S = 0, \tag{59}$$

considering S is independent of θ and Φ.

Therefore, we have

$$2\partial_v S\partial_r S + \left(1-\frac{2M}{r}-\phi_v^2\right)\left(\partial_r S\right)^2 = 0 \tag{60}$$

Now, let us choose the action S in the following form [29]

$$S(v,r) = -\int_0^v E(v')dv' + S_0(v,r), \tag{61}$$

so that

$$\partial_v S = -E(v) + \partial_v S_0 \text{ and } \partial_r S = \partial_r S_0.$$

Since S_0 is dependent on v and r, so we can write

$$\frac{dS_0}{dr} = \partial_r S_0 + \frac{dv}{dr}\partial_v S_0 = \partial_r S_0 + \frac{2}{F}\partial_v S_0, \tag{62}$$

where we have used $\frac{dv}{dr} = \frac{2}{F}$, F is defined in equation (23).

Now applying the equations (62) and (62) in equation (60), we get

$$F\frac{dS_0}{dr} = 2E(v), \tag{63}$$

since $\partial_r S_0 \neq 0$.

Therefore, we can obtain the solution of S_0 as

$$\begin{aligned} S_0 = 2E(v)\int\frac{dr}{F} &\equiv 2E(v)\int\frac{dr}{(1-\frac{2M}{r}-\phi_v^2)} \\ = \frac{2E(v)}{N}\int\frac{r\ dr}{r-2M/N} &= 2\pi i\frac{4ME(v)}{N^2}, \end{aligned} \tag{64}$$

with $N = 1-\phi_v^2$.

Here, we have used the Cauchy-integral formula since in the integral, r is analytic inside and on any simple closed contour C which is taken in the positive sense and $\frac{2M}{N}$ is a point interior to C. Therefore, (61) become,

$$S(v,r) = -\int_0^v E(v')dv' + 2\pi i\frac{4ME(v)}{N^2}. \tag{65}$$

So the wave function for the outgoing (and ingoing) massless particle can be read as

$$\Psi_{out}(v,r) = exp\Big[\frac{i}{\hbar}\Big(-\int_0^v E(v')dv' + \pi i\frac{4ME(v)}{N^2}\Big)\Big], \tag{66}$$

$$\Psi_{in}(v,r) = exp\Big[\frac{i}{\hbar}\Big(-\int_0^v E(v')dv' - \pi i\frac{4ME(v)}{N^2}\Big)\Big]. \tag{67}$$

The tunneling rate for the outgoing particle is

$$\Gamma \sim e^{-2ImS} \sim e^{-2\frac{4\pi ME(v)}{N^2}} = e^{-\frac{E(v)}{K_BT}}, \tag{68}$$

where K_B is Boltzman Constant.

Therefore, the Hawking temperature is

$$T_H = \frac{1}{8\pi K_B}\frac{N^2}{M} = \frac{1}{8\pi K_B}\frac{(1-\phi_v^2)^2}{M}. \tag{69}$$

If we consider $\phi_v^2 = 0$ and $m(v,r) = M(v)$, then we can get back the usual Hawking temperature for the Vaidya spacetime [30].

4. Conclusion

Based on Sawayama's [52] modified definition of the dynamical horizon, we have investigated the Hawking radiation in the **K**-essence emergent Vaidya spacetime. Manna et al. [18] have established the connection between the **K**-essence geometry and the Vaidya spacetime, and hence generated the new spacetime namely "the **K**-essence emergent Vaidya spacetime". We evaluate the dynamical horizon equation (51) for the spherically symmetric **K**-essence emergent Schwarzschild Vaidya spacetime which is different from the Sawayama's result, i.e., the usual Vaidya dynamical horizon equation.

From this study of the Hawking radiation using the dynamical horizon equation, we have arrived to the transcendental equation (56) which is far different from the Sawayama's [52] transcendental equation (38) and this equation can not be solved analytically in the presence of the Wright omega function [55] but in future one can find a numerical solution of this equation. So, in this case, we are not able to find that the behavior of the mass of the **K**-essence emergent Schwarzschild Vaidya black hole $m(v,r)$ (for fixed M) at the horizon from the relation $m(v,r) = M + \frac{r}{2}(1-N)$. On the other hand, we have also evaluated the Hawking temperature for the **K**-essence emergent Schwarzschild Vaidya metric (22) using tunneling mechanism. The Hawking temperature is $T_H = \frac{1}{8\pi K_B}\frac{(1-\phi_v^2)^2}{M}$, which is different from the usual Vaidya case.

We hasten to add, two of us (B.M. and G.M.) has coauthored papers [42–45], used the **K**-essence theory as a class of theoretical model of the dark energy. But here we use this theory from a purely gravitational standpoint, rather than the cosmological context of dark energy whose very existence is still not beyond doubt [58,59], based on the latest analysis of data from the Planck consortium [60].

References

1. S. Hawking, *Commun. Math. Phys.* **43** (1975) 199.
2. S. Hawking, *Phys. Rev. Lett.* **26** (1971) 1344.
3. L. Smarr, *Phys. Rev. Lett.* **71** (1973) 30.
4. J. Bardeen, B. Carter and S. Hawking, *Commun. Math. Phys.* **31** (1973) 161.
5. S. Hawking, *Nature* (London) **30** (1974) 248.
6. J. Bekenstein, *Phys. Rev. D* **7** (1973) 2333.
7. J. Bekenstein, *Phys. Rev. D* **9** (1974) 3292.
8. G. Gibbons and S. Hawking, *Phys. Rev. D* **15** (1977) 2738.
9. W.M. Unruh, *Phys. Rev. Lett.* **46** (1981) 1351.
10. S.W. Hawking, G.T. Horowitz and S.F. Ross, *Phys. Rev. D* **51** (1995) 4302.
11. P.C. Vaidya, *Phys. Rev.* **83** (1951) 1.
12. P.C. Vaidya and I.M. Pandya, *Prog. Theo. Phys.* **35** (1966) 1.
13. P.C. Vaidya, *Phys. Rev.* **174** (1968) 5.
14. P.C. Vaidya, *Gen. Rel. Gravit.* **31** (1999) 1.
15. P.C. Vaidya, *Astrophys. J.* **144** (1966) 3.
16. V. Husain, *Phys. Rev. D* **53** (1996) 4.
17. A. Wang and Y. Wu, *Gen. Rel. Gravit.* **31** (1999) 1.
18. G. Manna et. al., *Phys. Rev. D* **101** (2020) 124034.
19. G. Manna, *Eur. Phys. J. C* **80** (2020) 813.
20. M.K. Parikh and F. Wilczek, *Phys. Rev. Lett.* **85** (2000) 5042.
21. P. Mitra, *Phys. Lett. B* **648** (2007) 240.
22. B. Chatterjee, A. Ghosh and P. Mitra, *Phys. Lett. B* **661** (2008) 307.
23. B. Chatterjee and P. Mitra, *Phys. Lett. B* **675** (2009) 240.
24. P. Mitra, Report number: SINP/TNP/96-05, arXiv: 0902.2055.
25. K. Srinivasan and T. Padmanabhan, *Phys. Rev. D* **60** (1999) 024007.
26. S. Shankaranarayanan, T. Padmanabhan and K. Srinivasan, *Class. Quantum Gravit.* **19** (2002) 2671.
27. R. Kerner and R.B. Mann, *Phys. Rev. D* **73** (2006) 104010.
28. Y. Kuroda, *Prog. Theor. Phys.* **71** (1984) 1422.
29. H.M. Siahaan and Triyanta, *Int. J. Mod. Phys.* **A25** (2010) 145.
30. H. Tang et al., *Int. J. Theor. Phys* **57** (2018) 2145.
31. S. Wanglin et al., *Nucl. Phys. B* (Proc. Suppl.), **166** (2007) 270.
32. M. Visser, C. Barcelo and S. Liberati, *Gen. Rel. Gravit.* **34** (2002) 1719.
33. E. Babichev, V. Mukhanov and A. Vikman, *JHEP* **0609** (2006) 061.

34. E. Babichev, M. Mukhanov and A. Vikman, *JHEP* **0802** (2008) 101.
35. A. Vikman, *K-essence: Cosmology, causality and Emergent Geometry*, (Dissertation an der Fakultat fur Physik, Arnold Sommerfeld Center for Theoretical Physics, der Ludwig-Maximilians-Universitat Munchen, Munchen, den 29.08.2007).
36. E. Babichev, M. Mukhanov and A. Vikman, *Looking beyond the Horizon* (WSPC-Proceedings, October 23, 2008).
37. R.J. Scherrer, *Phys. Rev. Lett.* **93** (2004) 011301.
38. L.P. Chimento, *Phys. Rev. D* **69** (2004) 123517.
39. M. Born and L. Infeld, *Proc. Roy. Soc. Lond A* **144** (1934) 425.
40. W. Heisenberg, *Zeit. Phys.* **113** (1939) 61.
41. P.A.M. Dirac, *R. Soc. London Proc. Series A* **268** (1962) 57.
42. D. Gangopadhyay and G. Manna, *Euro. Phys. Lett.* **100** (2012) 49001.
43. G. Manna and D. Gangopadhyay, *Eur. Phys. J. C* **74** (2014) 2811.
44. G. Manna and B. Majumder, *Eur. Phys. J. C* **79** (2019) 553.
45. G. Manna et al., *Eur. Phys. J. Plus* **135** (2020) 107.
46. A. Ashtekar and B. Krishnan, *Phys. Rev. Lett.* **89** (2002) 26.
47. A. Ashtekar and B. Krishnan, *Phys. Rev. D* **68** (2003) 104030.
48. A. Ashtekar and G.J. Galloway, *Adv. Theor. Math. Phys.* **9** (2005) 1.
49. A. Ashtekar and B. Krishnan, *Liv. Rev. Rel.* **7** (2004) 10.
50. S.A. Hayward, *Phys. Rev. D* **49** (1994) 6467.
51. B. Krishnan, *Quasi-local Black Hole Horizons.* In: A. Ashtekar and V. Petkov (eds) Springer Handbook of Spacetime (Springer Handbooks. Springer, Berlin, Heidelberg. $https://doi.org/10.1007/978-3-642-41992-8_25$, 2014).
52. S. Sawayama, *Phys. Rev. D* **73** (2006) 064024.
53. S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).
54. P. Candelas, *Phys. Rev. D* **21** (1980) 2185.
55. R.M. Corless and D.J. Jeffrey, *Artificial Intelligence, Automated Reasoning, and Symbolic Computation* (AISC 2002, Calculemus 2002. Lecture Notes in Computer Science, vol 2385, Springer, Berlin, Heidelberg).
56. R.M. Corless et al., *Adv. in Comput. Math.* **5** (1996) 329.
57. R.M. Corless and D.J. Jeffrey, *Sigsam Bulletin* **30** (1996) 28.
58. J.T. Nielsen, A. Guffanti and S. Sarkar, *Sci. Rep.* **6** (2016) 3559.
59. J. Colin et al., *Astron. Astrophys.* **631** (2019) L13.
60. P.R. Ade et al., Planck Collaboration: Planck 2015 results. XX. Constraints on inflation, *Astron. Astrophys.* (Inflation-driver: September 2017).