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Modified Newtonian Gravity: Explaining observations of sub- and super-Chandrasekhar limiting mass white dwarfs

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Abstract: The idea of possible modification to gravity theory, whether it is in the Newtonian or general relativistic premises, is there for quite sometime. Based on it, astrophysical and cosmological problems are targeted to solve. But none of the Newtonian theories of modification has been performed from the first principle. Here, we modify Poisson's equation and propose two possible ways to modify the law gravitation which, however, reduces to Newton's law far away from the center of source. Based on these modified Newton's laws, we attempt to solve problems lying with white dwarfs. There are observational evidences for possible violation of the Chandrasekhar mass-limit significantly: it could be sub- as well as super-Chandrasekhar. We show that modified Newton's law, either by modifying LHS or RHS of Poisson's equation, can explain them.

Keywords: Newton's law; modified Poisson's equation; Chandrasekhar-limit; white dwarf

1. Introduction

Over the years, the researchers have explored the modifications to Einstein's gravity in order to explain astrophysical and cosmological data. However, any such modification proposed for the compact objects should be asymptotically flat. It should follow the reduction from modified Einstein's to Einstein's gravities and then to Newtonian gravity with distance from the source. Therefore, a modified Einstein's gravity may reduce to modified Newtonian gravity at some length scale.

In the present paper, we explore the possible modification to Poisson's equation to understand the possible modification to Newtonian gravity. Based on that, we target to resolve an astrophysical problem.

A white dwarf is a stellar core remnant composed mostly of electron-degenerate matter. The Chandrasekhar-limit is a theoretical limit for the maximum mass of a stable nonrotating and nonmagnetized white dwarf. If the mass of a white dwarf exceeds this limit, the force due to gravity becomes greater than that due to the electron degeneracy pressure. This leads to the star collapsing under its own gravity, leading to heating up of the plasma, which can result in a supernova. This is what happens in a type Ia supernova (SNIa).

A typical, slowly rotating, carbon-oxygen white dwarf accreting mass from a companion star explodes at a critical mass $\sim 1.4M_{\odot}$. Due to this, all type Ia supernovae (SNeIa) have a characteristic light curve, that is luminosity as a function of time. This makes SNeIa a "standard candle", which can be used to study the universe in various ways. Notably, observations of SNeIa led to the conclusion that the universe is undergoing an accelerated expansion.

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However, there are observations of highly over-luminous SNeIa, e.g., SN 2003fg, SN 2006gz, SN 2007if, SN 2009dc [1,2] with progenitor masses believed to be as high as $2.8M_{\odot}$ and highly under-luminous SNeIa, e.g., SN 1991bg, SN 1997cn, SN 1998de, SN 1999by, SN 2005bl with inferred progenitor masses being as low as $0.5\ M_{\odot}$ [3–5].

As a possible explanation of these anomalous SNeIa, Mukhopadhyay and his collaborators had earlier explored modifications to general relativistic gravity models and the resulting change in the Chandrasekhar mass-limit [6,7].

This work explores similar modifications to Newton's model of gravity to see if modifications to corresponding classical quantities also produce similar results.

2. The Modification to Newtonian gravity

The aim here is to have a formula that is able to reproduce Newtonian results in the weak field limit and mimic general relativistic results in the strong field limit. The original Poisson's equation for Newtonian gravitational potential $\phi(\mathbf{r})$ for a density distribution $\rho(\mathbf{r})$ is

$$\nabla^2\phi = 4\pi\rho G. \tag{1}$$

The following are two general types of modifications that are considered in this work:

$$\nabla^2\phi + A\nabla^4\phi = 4\pi\rho G \tag{2}$$

$$\nabla^2\phi = 4\pi G(\rho + B\rho^2 + ...). \tag{3}$$

3. LHS modification: general solution to the equation

The modified formula can be provided as

$$\nabla^2\phi + A\nabla^4\phi = 4\pi\rho G. \tag{4}$$

Using Green's method, with the constraint that $\phi \in \mathbb{R}$, we get the following expressions:

$$\text{For } A > 0\text{: } \phi(\mathbf{r}) = \int \frac{d^3\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|}\left(\int d^3\mathbf{r}''\frac{\cos\left(\frac{|\mathbf{r}'-\mathbf{r}''|}{A}\right)}{4\pi|\mathbf{r}'-\mathbf{r}''|}\frac{4\pi G\rho(\mathbf{r}'')}{A}\right), \tag{5}$$

$$\text{For } A < 0\text{:} \phi(\mathbf{r}) = \int \frac{d^3\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|}\left(\int d^3\mathbf{r}''\frac{\exp\left(-\frac{|\mathbf{r}'-\mathbf{r}''|}{\sqrt{|A|}}\right)}{4\pi|\mathbf{r}'-\mathbf{r}''|}\frac{4\pi G\rho(\mathbf{r}'')}{|A|}\right). \tag{6}$$

We have provided detailed explanation in Appendix A.

3.1 Calculation of the mass limit

Coming to the ideal white dwarf model which satisfies the equations:

$$P = K\rho^{1+\frac{1}{n}}, \tag{7}$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho, \tag{8}$$

$$\frac{\nabla P}{\rho} = -\nabla\phi, \tag{9}$$

$$\implies \frac{1}{r^2}\frac{d}{dr}(\frac{r^2}{\rho}\frac{dP}{dr}) = -\nabla^2\phi = -f(r), \tag{10}$$

where P is the pressure of white dwarf matter, ρ the density, M_r the mass enclosed in the radius r, n the polytropic index and K the polytropic constant.

Let θ be a dimensionless function of r so that

$$\rho(r) \equiv \rho = \rho_c \theta^n, \tag{11}$$

where ρ_c being the density at the centre of the white dwarf.

Similarly, considering dimensionless variable ξ such that

$$r = a\xi, \tag{12}$$

we obtain

$$-f(r) = \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = \frac{(1+n)K\rho_c^{1/n}}{a^2}\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right). \tag{13}$$

From the modified equation for gravity given by Eq. (4):

$$f(r) + A\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) = g(r) = 4\pi\rho_c\theta^n G, \tag{14}$$

$$\left(\frac{2}{\xi}\frac{d\theta}{d\xi} + \frac{d^2\theta}{d\xi^2}\right) + \frac{A}{a^2}\left(\frac{4}{\xi}\frac{d^3\theta}{d\xi^3} + \frac{d^4\theta}{d\xi^4}\right) = -\frac{a^2 4\pi G}{(1+n)K\rho_c^{\frac{1-n}{n}}}\theta^n. \tag{15}$$

If we let $a^2 = \frac{(1+n)K\rho_c^{\frac{1-n}{n}}}{4\pi G}$, then we obtain

$$\left(\frac{2}{\xi}\frac{d\theta}{d\xi} + \frac{d^2\theta}{d\xi^2}\right) + \frac{A}{a^2}\left(\frac{4}{\xi}\frac{d^3\theta}{d\xi^3} + \frac{d^4\theta}{d\xi^4}\right) = -\theta^n. \tag{16}$$

Notice that when $A = 0$, the above equation reduces to the Lane-Emden equation as expected.

For $n = 3$ (i.e. the relativistic case), we will numerically find the solution of $\theta(\xi)$, which will give us the mass-limit of a white dwarf as a function of A/a^2 based on

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi a^3 \rho_c \int_0^{\xi_1} \xi^2\theta^n d\xi. \tag{17}$$

3.2 Numerical solution of $\rho(r)$ for white dwarfs

Having added a new parameter to the Lane-Emden equations, we will need some extra information than what is usually needed to find numerical solutions in the Newtonian gravity. Fig. 1 shows the variation of M_{lim}/M_{Ch}, with M_{lim} being new mass-limit and M_{Ch} being original Chandrasekhar-limit, as a function of a^2/A for different values of the extra unknown constraint $\theta''(0)$ shown as labels.

Since 'a' depends on ρ_c and n, we can obtain different values of Chandrasekhar-limit for different ρ_c and even subtly different n, both of which are physical parameters. Hence given a fixed value of the modification parameter A which we naturally expect to be a universal constant, there can be conditions where the Chandrasekhar mass-limit differs depending on conditions inside the white dwarf.

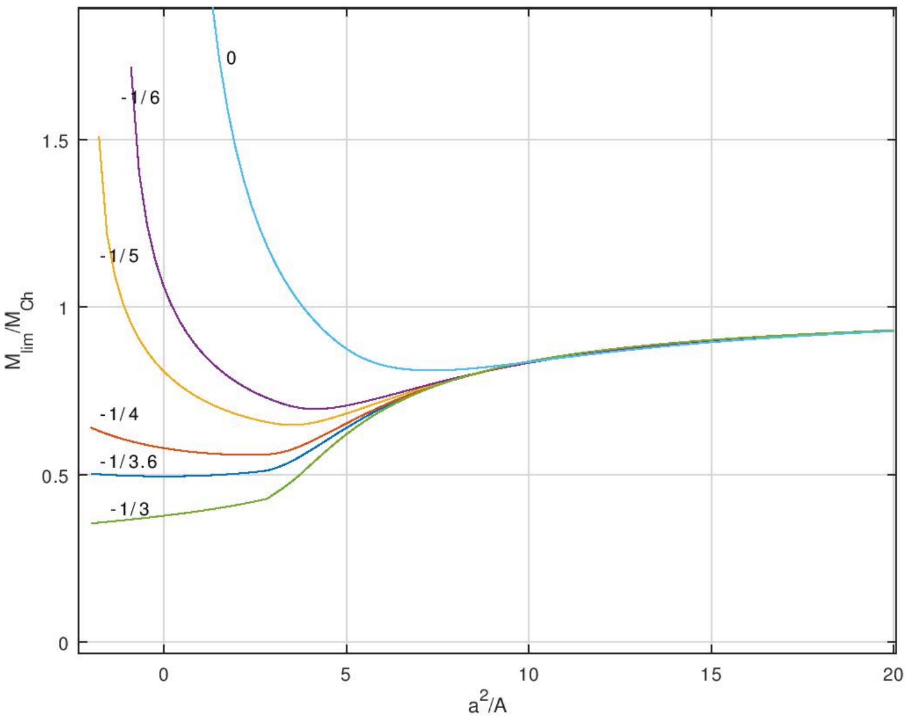

Fig. 1 Variation of M_{lim}/M_{Ch} with a^2/A for different $\theta''(0)$ shown as labels.

3.3 Analytical limits of M_{lim}/M_{Ch} as $A \to 0$ and $A \to \infty$

The modified Lane-Emden equation is

$$\left(\frac{2}{\xi}\frac{d\theta}{d\xi} + \frac{d^2\theta}{d\xi^2}\right) + \frac{A}{a^2}\left(\frac{4}{\xi}\frac{d^3\theta}{d\xi^3} + \frac{d^4\theta}{d\xi^4}\right) = -\theta^n. \tag{18}$$

Analytical limits can give us some verification of the correctness of numerical solutions. For example, the results in Fig. 1 appear to be correct at least in the $a^2/A \geq 0$ regime.

3.3.1 Limit $A \to 0$:

For $A/a^2 << 1$, assuming slight perturbation to the original Lane-Emden equation, the solution gives us:

$$\frac{M_{lim}}{M_{Ch}} = \frac{\int_0^{\xi_1} \xi^2 \theta^3 d\xi}{2.01824} = \left(1 + \frac{A}{a^2} \frac{2.1173}{2.01824}\right)\left(1 - 3\frac{A}{a^2}\right). \tag{19}$$

Fig. 2 confirms that the solution passes $A = 0$ point smoothly with $M_{lim}/M_{Ch} = 1$, confirming the correctness of results around $A = 0$.

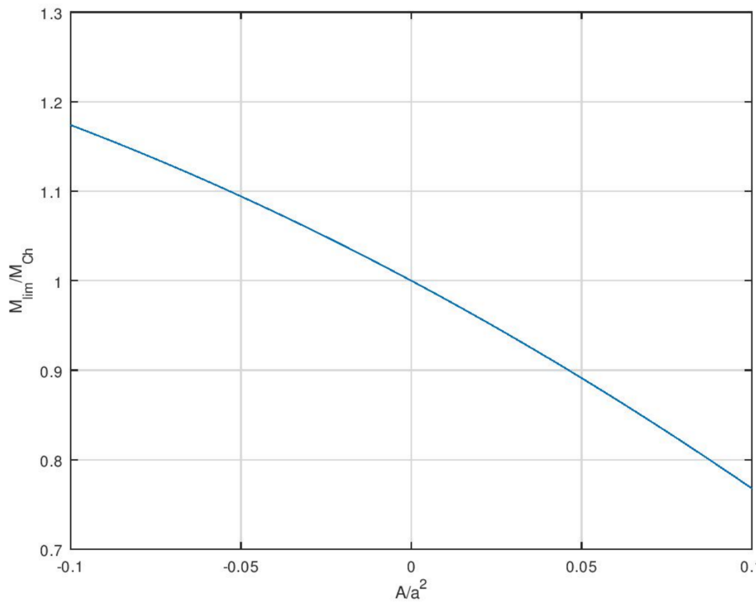

Fig. 2 Variation of M_{lim}/M_{Ch} with A/a^2 around $A = 0$.

3.3.2 Limit $A \to \infty$:

In this case the mass-limit depends on second derivative of $\theta(r) = [\rho(r)/\rho(0)]^{1/n}$ (where $\rho(0) = \rho_c$) at the centre of the white dwarf, and hence we obtain

$$\frac{M_{lim}}{M_{Ch}} = \frac{1}{15}\left(\frac{-2}{\theta''(0)}\right)^{3/2}. \tag{20}$$

3.4 Mass-radius relation

For the mass-radius relation, the density as a function of distance from center is numerically computed[3] using the following equations:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho, \tag{21}$$

$$\frac{\nabla P}{\rho} = -\nabla \phi. \tag{22}$$

Chandrasekhar's exact equation of state is given by

$$P = K_1 \left[x(2x^2 - 3)\sqrt{x^2+1} + 3\sinh^{-1} x \right], \tag{23}$$

$$\rho = K_2 x^3, \tag{24}$$

where $K_1 = 8\pi\mu_e m_H (m_e c)^3/3h^3$ and $K_2 = \pi m_e^4 c^5/3h^3$.

The modified gravity equation is:

$$\nabla^2\phi + A\nabla^4\phi = 4\pi\rho G. \tag{25}$$

Fig. 3 shows the variation radius of white dwarfs as a function of their mass.

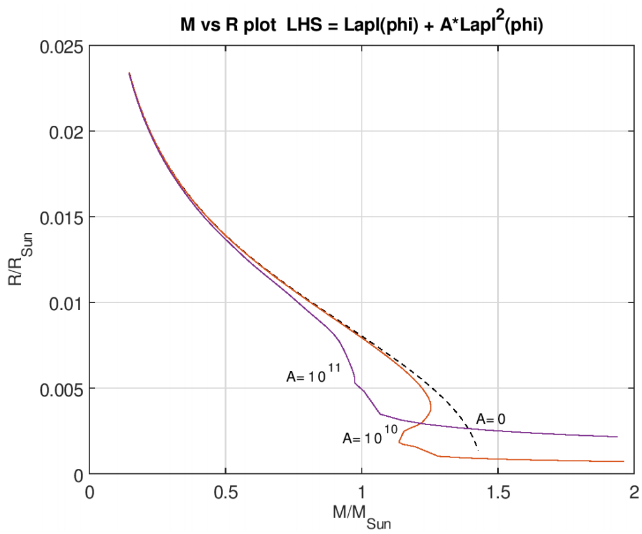

Fig. 3 Variation of radius with mass, where A is in units of m^2. The central density varies from $\rho_c = 10^8$ to $10^{13} kg/m^3$.

[3] Using GNU Octave's 'ode15s' function - a variable step, variable order method based on Backward Difference Formulas (BDF).

4. RHS modification: general solution to the equation

The modified formula in this case is

$$\nabla^2\phi = 4\pi G(\rho + B\rho^2 + ...). \tag{26}$$

Let $\rho_{\rm eff} = \rho + B\rho^2 + ...$

We explicitly know the solution of the Poisson equation $\nabla^2\phi = \rho_{\rm eff}(\mathbf{r})$ is

$$\phi(\mathbf{r}) = -\int \frac{\rho_{\rm eff}(\mathbf{r'})}{4\pi|\mathbf{r}-\mathbf{r'}|} d^3\mathbf{r'}. \tag{27}$$

Therefore, the form of gravitational potential will be exactly same as the usual Newtonian gravitational potentials, but with $\rho_{\rm eff}$ instead of ρ.

4.1 Mass-radius relation

Using the same method as in Sec. 3, with the new modified gravity equation, the variation of radius for white dwarfs as a function of their mass is generated numerically[4].

The modified equation considered is

$$\nabla^2\phi = 4\pi G(\rho + A\rho^2). \tag{28}$$

One can observe in Fig. 4 that when $\rho_{\rm eff} > \rho$, the mass is lower for a given radius, as expected, and likewise for $\rho_{\rm eff} < \rho$, the mass is higher than usual white dwarf mass, obtained based on Newton's law, for the same radius. Therefore, using higher order polynomial terms, at different densities, the white dwarf can show arbitrarily small or large masses in the range of ρ where the respective terms dominate.

5. Conclusion

Newton's law is a remarkably successful physics, well tested in laboratory, also is remarkably successful in explaining low energy physics. Several astrophysical features are also quite abide it. In this connection, the Chandrasekhar-limit perhaps is one of the most celebrated astrophysical discoveries in the 20th century, whose physical insight can be well understood in the Newtonian framework itself. However, observations of several peculiar over- and under-luminous SNeIa for about last three decades argue for the significant violation of the Chandrasekhar mass-limit. We have shown here that appropriate modifications to Poisson's equation and, hence, Newton's law can explain the significant violation of the Chandrasekhar-limit, as inferred from observations. It argues that while the existence of the Chandrasekhar-limit is sacrosanct, its value need not be. We expect the proposed modifications to Poisson's equation and modified Newton's law, which reduces to Newton's law asymptotically, to have far reaching implications.

[4] Using GNU Octave's 'ode15s' function - a variable step, variable order method based on Backward Difference Formulas (BDF).

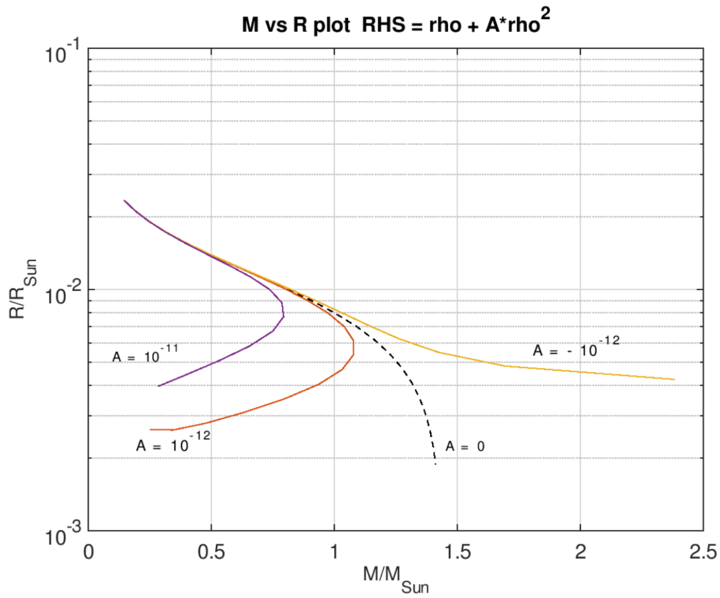

Fig. 4 Variation of radius with mass, where A is in units of m^3/kg. The central density varies from $\rho_c = 10^8$ to $10^{11} kg/m^3$.

Appendix A: General solution of the equation with modified LHS

The modified equation is given by

$$\nabla^2\phi + A\nabla^4\phi = 4\pi\rho G. \tag{29}$$

$$\text{Let } u(\mathbf{r}) = \nabla^2\phi(\mathbf{r}),$$
$$\implies \left[A\nabla^2 + 1\right] u(\mathbf{r}) = 4\pi\rho G. \tag{30}$$

From the solution of Screened Poisson Equation (derived using Green's functions)

$$\left[\nabla^2 - \lambda^2\right] u(\mathbf{r}) = -f(\mathbf{r}), \tag{31}$$

$$\implies u(\mathbf{r}) = \int d^3\mathbf{r'} \frac{e^{-\sqrt{\lambda^2}|\mathbf{r}-\mathbf{r'}|}}{4\pi|\mathbf{r}-\mathbf{r'}|} f(\mathbf{r'}). \tag{32}$$

For $A < 0$, $\lambda^2 = \frac{1}{|A|}, f = \frac{4\pi\rho G}{|A|},$

$$u(r \to \infty) \to 0 \implies \lambda = \frac{1}{\sqrt{|A|}}.$$

Hence

$$u(\mathbf{r}) = \nabla^2\phi(\mathbf{r}) \implies \phi(\mathbf{r}) = \int \frac{d^3\mathbf{r'}}{4\pi|\mathbf{r}-\mathbf{r'}|} u(\mathbf{r'}), \tag{33}$$

$$\phi(\mathbf{r}) = \int \frac{d^3\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|}\left(\int d^3\mathbf{r}'' \frac{\exp\left(-\frac{|\mathbf{r}'-\mathbf{r}''|}{\sqrt{|A|}}\right)}{4\pi|\mathbf{r}'-\mathbf{r}''|}\frac{4\pi G\rho(\mathbf{r}'')}{|A|}\right). \tag{34}$$

Similarly, for $A > 0$, $\lambda^2 = -\frac{1}{A}, f = -\frac{4\pi\rho G}{A}$,

$$u(r) \in \mathbb{R} \implies e^{-\lambda|\mathbf{r}-\mathbf{r}'|} \equiv \frac{1}{2}\left(e^{\frac{-\iota}{\sqrt{A}}|\mathbf{r}-\mathbf{r}'|} + e^{\frac{\iota}{\sqrt{A}}|\mathbf{r}-\mathbf{r}'|}\right) \equiv \cos\left(\frac{1}{\sqrt{A}}|\mathbf{r}-\mathbf{r}'|\right).$$

Therefore

$$\phi(\mathbf{r}) = \int \frac{d^3\mathbf{r}'}{4\pi|\mathbf{r}-\mathbf{r}'|}\left(\int d^3\mathbf{r}'' \frac{\cos\left(\frac{|\mathbf{r}'-\mathbf{r}''|}{A}\right)}{4\pi|\mathbf{r}'-\mathbf{r}''|}\frac{4\pi G\rho(\mathbf{r}'')}{A}\right). \tag{35}$$

References

1. D.A. Howell et al., *Nature* **443** (2006) 308.
2. R.A. Scalzo et al., *ApJ* **713** (2010) 1073.
3. A.V. Filippenko et al., *AJ* **104** (1992) 1543.
4. P.A. Mazzali et al., *Mon. Not. R. Astron. Soc.* **284** (1997) 151.
5. S. González-Gaitán et al., *ApJ* **727** (2011) 107.
6. U. Das and B. Mukhopadhyay, *JCAP* **5** (2015) 045.
7. S. Kalita and B. Mukhopadhyay, *JCAP* **9** (2018) 007.