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Factorization of the metric tensor in Kerr geometry

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Abstract: In Kerr spacetime, we use a null tetrad to factor the metric tensor in Boyer-Lindquist coordinates to construct a generator of the Lanczos potential for the conformal tensor.

Keywords: Kerr metric, Lanczos potential, Weyl tensor, Newman-Penrose formalism.

1. Introduction

Einstein's general relativity, which was proposed in 1915 for accounting gravitational effect on spacetime geometry, has been successful for demonstrating several phenomena related to astrophysics as well as cosmology. Basically, the vacuum solution of the Einstein field equations by Schwarzschild in the same year received an instant impact which however was based on the simplest assumption of static configuration and thus lacks of physical reality of rotation. Much later, in the year 1963, Kerr [1] introduced a rotation to the spherical system and thus provided a generalized solution to the Schwarzschild spacetime.

A good amount of subsequent works on the Kerr metric are available in literature [1 – 7] in connection to its different mathematical as well as physical structures and properties. On the other hand, an efficient mathematical tool is also available as background of the Kerr metric which is known as the Newman-Penrose formalism [8] which has been followed by several interesting works [9 – 18].

In the present investigation we use a null tetrad in the Kerr spacetime to factor the metric tensor in Boyer-Lindquist coordinates to construct a generator of the Lanczos potential for the conformal tensor. This has been carried out in Sec. 2 whereas we present some relevant comments on the result of our investigation in Sec. 3.

2. Mathematical Derivation

We shall employ the Newman-Penrose (NP) formalism [8 -- 18] with the null tetrad $(l^\mu, n^\mu, m^\mu, \overline{m}^\mu)$ such that $l^\alpha\, n_\alpha = -\, m^\alpha\, \overline{m}_\alpha = 1$, then any metric tensor can be factored in the following form:

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$$g_{\alpha\beta} = l_\alpha n_\beta + l_\beta n_\alpha - m_\alpha \bar{m}_\beta - m_\beta \bar{m}_\alpha = 16\, P_\alpha{}^\mu P_{\mu\beta}\,, \tag{1}$$

where

$$P_\alpha{}^\mu = \frac{1}{4}(m_\alpha m^\mu + \bar{m}_\alpha \bar{m}^\mu - l_\alpha l^\mu - n_\alpha n^\mu), \tag{2}$$

which can be applied to the Kerr metric [1 -- 7] in Boyer-Lindquist coordinates (t, r, θ, φ) [10, 11, 13, 19 -- 22]:

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{\Sigma}{C}\, dr^2 - \Sigma\, d\theta^2 + \frac{4Mar}{\Sigma} \sin^2\theta\, dt\, d\varphi - \left(r^2 + a^2 + \frac{2Ma^2 r}{\Sigma} \sin^2\theta\right) \sin^2\theta\, d\varphi^2,$$
$$A = r - i\, a\cos\theta\,, \qquad \Sigma = A\bar{A} = r^2 + a^2 \cos^2\theta, \qquad C = r^2 - 2M\, r + a^2. \tag{3}$$

The interesting thing is that if we use the null tetrad:

$$(l^\alpha) = \frac{1}{\sqrt{2\Sigma C}}\,(r^2 + a^2, -C, 0, a), \qquad (n^\alpha) = \frac{1}{\sqrt{2\Sigma C}}\,(r^2 + a^2, C, 0, a),$$
$$(m^\alpha) = \frac{1}{\sqrt{2\Sigma}}\left(a \sin\theta,\ 0, -i, \frac{1}{\sin\theta}\right), \qquad (m^\alpha) = \frac{1}{\sqrt{2\Sigma}}\left(a \sin\theta,\ 0, i, \frac{1}{\sin\theta}\right), \tag{4}$$

then (2) acquires the structure:

$$(P_\mu{}^\alpha) = (P^\alpha{}_\mu) = \frac{1}{4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad (P_{\alpha\nu}) = (P_{\nu\alpha}) = \frac{1}{4}\begin{pmatrix} -g_{00} & 0 & 0 & -g_{03} \\ 0 & g_{11} & 0 & 0 \\ 0 & 0 & g_{22} & 0 \\ -g_{03} & 0 & 0 & g_{33} \end{pmatrix}, \tag{5}$$

and (5) turns out to be a generator of the Lanczos potential $K_{\mu\nu\alpha}$ [23 -- 29] for the Weyl tensor $C_{\mu\nu\alpha\beta}$:

$$K_{\mu\nu\alpha} = P_{\alpha\nu;\mu} - P_{\alpha\mu;\nu} + g_{\alpha\nu} A_\mu - g_{\alpha\mu} A_\nu\,, \quad (A_\mu) = \left(\frac{1}{3} P^\nu{}_{\mu;\nu}\right) = \left(0, \frac{r-M}{6\,C}, \frac{ctg\,\theta}{6}, 0\right),$$
$$K_{\mu\nu\alpha} = -K_{\nu\mu\alpha}\,, \qquad K_{\mu\nu\alpha} + K_{\nu\alpha\mu} + K_{\alpha\mu\nu} = 0, \tag{6}$$

verifying the Lanczos gauges [23]:

$$K^{\mu\nu}{}_\nu = 0, \qquad K^{\mu\nu\alpha}{}_{;\alpha} = 0, \tag{7}$$

hence our potential (6) satisfies the Lanczos-Illge wave equation [30 -- 35] in this empty spacetime:

$$\Box\, K_{\mu\nu\alpha} = 0. \tag{8}$$

The corresponding conformal tensor is generated via the expression:

$$C_{\mu\nu\alpha\beta} = K_{\mu\nu\alpha;\beta} - K_{\mu\nu\beta;\alpha} + K_{\alpha\beta\mu;\nu} - K_{\alpha\beta\nu;\mu} + K_{\nu\alpha}\, g_{\mu\beta} - K_{\nu\beta}\, g_{\mu\alpha} + K_{\mu\beta}\, g_{\nu\alpha} - K_{\mu\alpha}\, g_{\nu\beta}\,,$$
$$K_{\mu\nu} = K_{\nu\mu} = K_{\mu\alpha\nu}{}^{;\alpha}\,, \tag{9}$$

which has type D in the Petrov classification [10, 11, 13, 27, 36].

The canonical null tetrad [37 -- 38] given in (4) implies the following spin coefficients:

$$\kappa=\sigma=\lambda=\nu=0, \qquad \rho=\mu=\frac{1}{A}\sqrt{\frac{C}{2\Sigma}}, \qquad \pi=\tau=-\frac{a\sin\theta}{A\sqrt{2\Sigma}}, \qquad \alpha=\beta=-\frac{a+i\,r\cos\theta}{2A\sqrt{2\Sigma}\,\sin\theta},$$
$$\varepsilon=\gamma=\frac{a^2-M\,r+i\,a\,(r-M)\cos\theta}{2\,A\sqrt{2\Sigma C}}, \quad \pi+\bar{\pi}=2(\beta+\bar{\beta}), \quad \rho-\bar{\rho}=2(\varepsilon-\bar{\varepsilon}), \quad \psi_2=-\frac{M}{A^3}=4(\varepsilon\rho-\pi\beta), \tag{10}$$

thus (6) can be written in terms of the null tetrad:

$$K_{abc}+i\ {}^*K_{abc}=2\,[\Omega_0\,(U_{ab}\,n_c+V_{ab}\,l_c)+\Omega_1\,(M_{ab}\,(l_c+n_c)-U_{ab}\,m_c-V_{ab}\,\bar{m}_c)+$$
$$+\,\Omega_2\big(V_{ab}\,n_c+U_{ab}\,l_c-M_{ab}\,(m_c+\bar{m}_c)\big)-\Omega_3\,(V_{ab}\,m_c+U_{ab}\,\bar{m}_c)], \tag{11}$$
$$M_{ab}=m_a\,x\,\bar{m}_b+n_a\,x\,l_b\,, \qquad U_{ab}=\bar{m}_a\,x\,n_b\,, \qquad V_{ab}=l_a\,x\,m_b\,,$$

with the NP components of the Lanczos potential:

$$\Omega_0=\Omega_7=-\frac{\pi}{4}, \quad \Omega_1=\Omega_6=-\left(\frac{\varepsilon}{3}+\frac{\rho}{12}\right), \quad \Omega_2=\Omega_5=-\left(\frac{\beta}{3}+\frac{\pi}{12}\right), \quad \Omega_3=\Omega_4=-\frac{\rho}{4}, \tag{12}$$

verifying (9), that is, the Weyl-Lanczos equations [17, 26 -- 29, 33, 39]:

$$\psi_0=2[\delta\Omega_0-D\Omega_4+(-\bar{\alpha}-3\beta+\bar{\pi})\Omega_0+3\sigma\Omega_1+(\bar{\rho}+3\varepsilon-\bar{\varepsilon})\Omega_4-3\kappa\Omega_5],$$
$$2\psi_1=\Delta\Omega_0+3\delta\Omega_1-\bar{\delta}\Omega_4-3D\Omega_5-(3\gamma+\bar{\gamma}+3\mu-\bar{\mu})\Omega_0+3(-\bar{\alpha}-\beta+\bar{\pi}+\tau)\Omega_1+6\sigma\Omega_2+$$
$$+(3\alpha-\bar{\beta}+3\pi+\bar{\tau})\Omega_4+3(\varepsilon-\bar{\varepsilon}-\rho+\bar{\rho})\Omega_5-6\kappa\Omega_6,$$
$$\psi_2=\Delta\Omega_1+\delta\Omega_2-\bar{\delta}\Omega_5-D\Omega_6-\nu\Omega_0-(2\mu-\bar{\mu}+\gamma+\bar{\gamma})\Omega_1+(-\bar{\alpha}+\beta+\bar{\pi}+2\tau)\Omega_2+\sigma\Omega_3+$$
$$+\lambda\Omega_4+(\alpha-\bar{\beta}+2\pi+\bar{\tau})\Omega_5-(\varepsilon+\bar{\varepsilon}-\bar{\rho}+2\rho)\Omega_6-\kappa\Omega_7, \tag{13}$$
$$2\psi_3=3\Delta\Omega_2+\delta\Omega_3-3\bar{\delta}\Omega_6-D\Omega_7-6\nu\Omega_1+3(\bar{\mu}-\mu-\bar{\gamma}+\gamma)\Omega_2+(-\bar{\alpha}+3\beta+3\tau+\bar{\pi})\Omega_3+$$
$$+6\lambda\Omega_5+3(-\alpha-\bar{\beta}+\pi+\bar{\tau})\Omega_6-(3\varepsilon+\bar{\varepsilon}-\bar{\rho}+3\rho)\Omega_7,$$
$$\psi_4=2[\Delta\Omega_3-\bar{\delta}\Omega_7-3\nu\Omega_2+(\bar{\mu}+3\gamma-\bar{\gamma})\Omega_3+3\lambda\Omega_6+(-3\alpha-\bar{\beta}+\bar{\tau})\Omega_7],$$

and also the NP version of the Lanczos differential gauge (7):

$$\Delta\Omega_2 - \delta\Omega_3 - \bar{\delta}\Omega_6 + D\Omega_7 - 2\nu\Omega_1 + (3\mu + \bar{\mu} + \gamma - \bar{\gamma})\Omega_2 + (\bar{\alpha} - 3\beta + \tau - \bar{\pi})\Omega_3 + 2\lambda\Omega_5 +$$
$$+ (-\alpha - \bar{\beta} + \bar{\tau} - 3\pi)\Omega_6 + (3\varepsilon + \bar{\varepsilon} - \rho - \bar{\rho})\Omega_7 = 0\,,$$

$$\Delta\Omega_0 - \delta\Omega_1 - \bar{\delta}\Omega_4 + D\Omega_5 + (\mu + \bar{\mu} - 3\gamma - \bar{\gamma})\Omega_0 + (\bar{\alpha} + \beta - \bar{\pi} + 3\tau)\Omega_1 - 2\sigma\Omega_2 +$$
$$+ (3\alpha - \bar{\beta} - \pi + \bar{\tau})\Omega_4 + (\bar{\varepsilon} - \varepsilon - \bar{\rho} - 3\rho)\Omega_5 + 2\kappa\Omega_6 = 0\,, \tag{14}$$

$$-\Delta\Omega_1 + \delta\Omega_2 + \bar{\delta}\Omega_5 - D\Omega_6 + \nu\Omega_0 + (\gamma + \bar{\gamma} - 2\mu - \bar{\mu})\Omega_1 + (-\bar{\alpha} + \beta + \bar{\pi} - 2\tau)\Omega_2 + \sigma\Omega_3 -$$
$$-\lambda\Omega_4 + (-\alpha + \bar{\beta} + 2\pi - \bar{\tau})\Omega_5 + (-\varepsilon - \bar{\varepsilon} + 2\rho + \bar{\rho})\Omega_6 - \kappa\Omega_7 = 0\,.$$

The factorization (1) gave the generator (2) of the Lanczos spin tensor, which allowed obtaining the vector (6):

$$(A_\mu) = \left(\tfrac{1}{3}\, P^{\nu}{}_{\mu;\nu}\right) = \left(0, \tfrac{r-M}{6\,C}, \tfrac{ctg\,\theta}{6}, 0\right) = (f_{,\mu}), \qquad f = \tfrac{1}{12}\, Ln\,(C\, sin^2\theta), \qquad \Box f = 0, \tag{15}$$

so it is interesting to see that the following function:

$$F = Ln\,\psi_2^{1/6} + 3\,f, \tag{16}$$

generates four spin coefficients:

$$\varepsilon = DF, \qquad \gamma = -\Delta F, \qquad \beta = \delta F, \qquad \alpha = -\bar{\delta} F, \tag{17}$$

in accordance with Nerozzi-Elbracht [40].

We note the Bianchi identities [10, 11, 13, 17, 27, 39, 41]:

$$\rho = DG\,, \qquad \mu = -\Delta G\,, \qquad \tau = \delta G\,, \qquad \pi = -\bar{\delta} G, \qquad G = Ln\,\psi_2^{1/3}\,, \tag{18}$$

and the relations:

$$Df = -\Delta f = \tfrac{M-r}{6\sqrt{2\Sigma C}} = \tfrac{1}{6}(2\varepsilon - \rho), \qquad \delta f = -\bar{\delta} f = -\tfrac{i\,ctg\,\theta}{6\sqrt{2\Sigma}} = \tfrac{1}{6}(2\beta - \pi). \tag{19}$$

We can use (5), (6) and the Christoffel symbols for the Kerr metric [42] [or to apply (4), (10) and (12) in (11)] to obtain the non-zero components of the Lanczos potential:

$$K_{abc} = \tfrac{1}{2}\,(g_{0c}\,\Gamma^0{}_{ab} + g_{3c}\,\Gamma^3{}_{ab}) - g_{ac}\,A_b\,, \qquad a, c = 0, 3, \qquad b = 1, 2,$$
$$K_{121} = \tfrac{\Sigma}{6\,C}\, ctg\,\theta, \qquad K_{122} = \tfrac{\Sigma\,(M-r)}{6\,C}\,, \tag{20}$$

that is:

$$K_{b00} = -\tfrac{1}{4}g_{00,b} + g_{00}A_b,\;\; K_{b03} = K_{b30} = -\tfrac{1}{4}g_{03,b} + g_{03}A_b,\;\; K_{b33} = -\tfrac{1}{4}g_{33,b} + g_{33}A_b,\quad b = 1, 2,$$
$$K_{211} = g_{11}\, A_2\,,\qquad K_{122} = g_{22}\, A_1\,, \tag{21}$$

where we observe that $K_{03\mu} = 0,\ \forall\, \mu$.

3. Conclusion

To summarize, in the present work we have shown derivation of the Lanczos potential through factorization of the metric tensor in Kerr geometry on purely analytical basis. The possible physical meaning of the Lanczos potential in general relativity, is an open problem. We believe that $K_{\mu\nu\alpha}$ behaves in the asymptotic region as a density for the angular momentum of a rotating black hole. It would be therefore justified if one could find out a few applications of this result. Though at this moment we have no specific applications of our result, however, we would like to mention that in a future project it is possible to study the importance of Lanczos potential in astrophysical as well as cosmological models with angular momentum, i.e., with rotation as in Kerr's black hole.

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