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# Cosmological Models with scale factors in $f(T)$ gravity

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**Abstract:** In this work, we have studied the cosmological model framed in an isotropic background in the $f(T)$ theory of gravity. The field equations are derived and the dynamical parameters are studied with two different type of scale factors that favours early deceleration and late time cosmic acceleration. The model is showing an accelerating behaviour which can be confronted from the behaviour of geometrical parameters. We also analysed the violation of null energy condition and strong energy condition. To study the dynamics of the universe cosmographic parameters has been investigated.

**Keywords:** Torsion Scalar; Cosmographic parameter; Energy Conditions

## 1 Introduction

In contemporary cosmology, the study of late-time cosmic acceleration events has been a significant focus. Supernovae of type Ia give strong evidence that the Universe expands at lightning speed. Cosmography is an appropriate method for investigating the cosmic expansion history in an almost model-independent approach, based on the hypothesis that the Universe is homogeneous and isotropic on large scales. Theoretical research and cosmological measurements of the Universe show that the Universe went through an inflationary phase at the beginning and an accelerated degree after that. It is theoretically possible to accomplish this in two ways. The content of the Universe is altered in the first method by adding new fields such as phantom scalars, canonical scalars, vector fields, etc. [1,2]. Modifying the gravitational sector [3] is the second technique. Teleparallel gravity [4,5] is a gravity theory that describes gravitational effects in terms of torsion rather than curvature, using the curvature-free Weitzenböck connection [6] to define the covariant derivative instead of the conventional torsionless Levi-Civita connection of general relativity (GR). It is comparable to GR in its most basic form, but it has a distinct physical meaning [5].

The Levi-Civita connection in GR denotes curvature but no torsion, whereas the Weitzenböck connection in teleparallelism implies torsion but no curvature [6]. The dynamical objects in this framework are the four linearly independent tetrad fields that provide the orthogonal basis for the tangent space at each point of space-time. The torsion tensor is also made up of the first derivative products of tetrad. A plethora of observations during the last two decades have confirmed the

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late-time cosmic acceleration of the Universe. These observations have developed a curiosity among the cosmologists to explain this late time dynamics. GR on its own fails to explain this expansion. Hence the idea of modifying GR has taken momentum in the last decade. Researcher are motivated to change either geometrical part of the field equation or the matter part. In $f(R)$ gravity theory, the scalar curvature $R$ in the Einstein-Hilbert action is changed to a suitable function $f(R)$. In another modified gravity theory known as teleparallel gravity, instead of curvature, torsion represents the gravitational interaction. Further, a generalization to teleparallel gravity (similar to $f(R)$) has been developed [7,8,9,10,11,12,13,14, 15] by replacing torsion scalar $T$ to a generic function $f(T)$. This modified gravity theory is termed as $f(T)$ gravity theory and Linder coined the name.

Moreover, there are two important differences between these two theories first one is the field equations in $f(T)$ gravity theory remain second order while one has fourth order equations in $f(R)$ and second theory is $f(T)$ gravity theory does not satisfy local Lorentz invariance (which is satisfied by $f(R)$ gravity) so that all 16 components of the vierbien are independent and hence it is not possible to fix six of them by a gauge choice [16]. The scalar perturbation technique is used to create the perturbed evolution equations, and the stability of the models is demonstrated in teleparallel gravity [17]. The vierbeins are parallel vector fields, which give the theory the descriptor teleparallel. The advantage is that the torsion tensor is formed solely from products of first derivatives of the tetrad. The presence of some exotic energy known as dark energy results from the change in the matter portion, and the difference in the geometric part results in extended gravity. The addition of dark energy to Einstein's equations as continuous stress has helped explain this expansion. The most exciting aspect is that, while we do not know the exact nature or origin of this energy, cosmologists agree on what it is not. Recent Planck data estimates a lion share of 68.3% in favour of dark energy. The late time cosmic dynamics and the consequent dark energy is understood through a dark energy equation of state parameter $\omega = \frac{p}{\rho}$, where p and $\rho$ respectively denote the pressure and energy density of dark energy. According to observations, the equation of state give value =-1 at present time. More generally, the expansion of the universe is accelerating for any equation of state $\omega < \frac{-1}{3}$. Several data sources, including the Pantheon supernovae sample, Hubble constant measurements cosmic microwave shift parameter, and redshift-space distortion measurement, have been used to restrict $f(T)$ gravity [18]. Pati et al. [19] have shown the cosmological models with LR, BR and PR Scenarios in the non-metricity gravity. In Ref. [20], the impact of violating the equivalence principle in the electromagnetic domain on $f(T)$ gravity is explored.

## 2 $f(T)$ Teleparallel Gravity

A particular modified theory of gravity which has attracted the interests of cosmologists is so-called $f(T)$ teleparallel gravity. Inspired by the formulation of $f(R)$ gravity, in which the Lagrangian of the gravitational field equations is a function, $f$, of the Ricci scalar $R$ of the underlying geometry, $f(T)$ gravity is a similar generalization. The associated dynamical fields are the four linearly independent vierbeins, and $T$ being connected to the antisymmetric connection resulting from the nonholonomic basis.

The action of $f(T)$ gravity is

$$S_{f(T)} = \frac{1}{16\pi G}\int d^4xe\,(f(T)) + S_m, \tag{1}$$

in which $e = det(e^i_\nu) = \sqrt{-g}$

In the holonomic frame the space time has the line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{2}$$

where $a = a(t)$ be the cosmological scale factor.

For this frame the torsion scalar, which depends on the signature of the metric, is

$$T = 6\left(\frac{\dot{a}}{a}\right)^2 = 6H^2, \tag{3}$$

where $H$ is the Hubble parameter.

We take $f(T)$ as

$$f(T) = T + \alpha(-T)^n. \tag{4}$$

For this model we obtain field equations as

$$12H^2 f_T(T) + f(T) = 16\pi G\rho, \tag{5}$$
$$48H^2\dot{H}f_{TT}(T) - 4(\dot{H} + 3H^2)f_T(T) - f(T) = 16\pi Gp, \tag{6}$$

where $\rho$ is the dark energy density and $p$ is the pressure of the dark energy.

From Eqs. (4), (5) and (6) we obtain the pressure, energy density and the equation of state parameter $\omega$ in terms of Hubble parameter as

$$\rho = \frac{6H^2 - (2n-1)\alpha(6H^2)^n}{16G\pi}, \tag{7}$$
$$p = \frac{-4\dot{H} - 6H^2 + \alpha(2n-1)(6H^2)^{n-1}(4n\dot{H} + 6H^2)}{16G\pi}, \tag{8}$$
$$\omega = \frac{-4\dot{H} - 6H^2 + \alpha(2n-1)(6H^2)^{n-1}(4n\dot{H} + 6H^2)}{6H^2 - (2n-1)\alpha(6H^2)^n}. \tag{9}$$

## 3 The Cosmological Models

In order to understand the background cosmology, we need to incorporate the scale factor to obtain the solution to the field equations. We will consider two different Hubble parameters and derive their respective pressure, energy density,equation of state parameter, deceleration parameter, snap parameter and jerk parameter.

To handle Eqs. (7) and (8), which are highly non-linear, we assume the hybrid scale factor (HSF), $a(t) = t^\mu e^{\nu t}$, such that $H = \nu + \frac{\mu}{t}$, where $\mu$ and $\nu$ are the arbitrary parameters and can be constrained in the ranges $\nu > 0$ and $0 < \mu < 1$ [21, 22, 24, 23]

The Hubble parameter we consider to analyze the dynamics of the Universe are

$$H = \frac{1}{t(2\lambda - t)(4\lambda - t)}, \tag{10}$$

$$H = \nu + \frac{\mu}{t}. \tag{11}$$

We consider relation of parameters with respect to redshift parameter $z$ to do our analysis. We are using $a(t) = \frac{1}{1+z}$ to get the desired relation.

For Eq. (10) we get $t$ in terms of $z$ as

$$t = 2\lambda - \frac{2\lambda(1+z)^{4\lambda^2}}{\sqrt{1+(1+z)^{8\lambda^2}}}, \tag{12}$$

and for Eq. (11) we get $t$ in term of $z$ as

$$t = \frac{\mu}{\nu} * ProductLog[\frac{(z+1)^{\frac{-1}{\mu}} * \nu}{\mu}]. \tag{13}$$

### 3.1 Scale Factor $a(t)$

We are using FLRW line element to obtain scale factor from Hubble parameter and for Eq. (10) we obtain scale factor as

$$a(t) = (t)^{\frac{1}{8\lambda^2}}(t - 4\lambda)^{\frac{1}{8\lambda^2}}(t - 2\lambda)^{\frac{-1}{4\lambda^2}}. \tag{14}$$

For Eq. (11) we obtain scale factor as

$$a(t) = t^{\mu}e^{\nu t}. \tag{15}$$

The kinematic property is universal, making it easy to explain the expansion of the cosmos, but the dynamic property is model-dependent. We will use a kinematic approach in this case. We used the FLRW space-time, which is homogeneous and isotropic, to model development of the Universe. The scale factor is used to characterize rate of expansion of the universe.

### 3.2 Deceleration Parameter $(q)$

To obtain $q$ we use the relation $q = \frac{-1}{a}\frac{d^2a}{dt^2}H^{-2}$ which for Eq. (10) yields

$$q = -1 + 3t^2 + 8\lambda^2 - 12\lambda t, \tag{16}$$

and for Eq. (11) we obtain $q$ as

$$q = -1 + \frac{\mu}{(\nu t + \mu)^2}. \tag{17}$$

We use Eqs. (16) and (17) to plot relation between $q$ and redshift parameter. The values of $\nu$ and $\mu$ in Eq. (11) are taken in such a way that it shows deceleration in the initial cosmic time and then acceleration in the later cosmic time so that it overlaps with the observations made till now. Value of $\lambda$ in Eq. (10) is taken as 0.5 value of $\nu$ and $\mu$ in Eq. (11) is taken as 0.585 and 0.2 respectively.

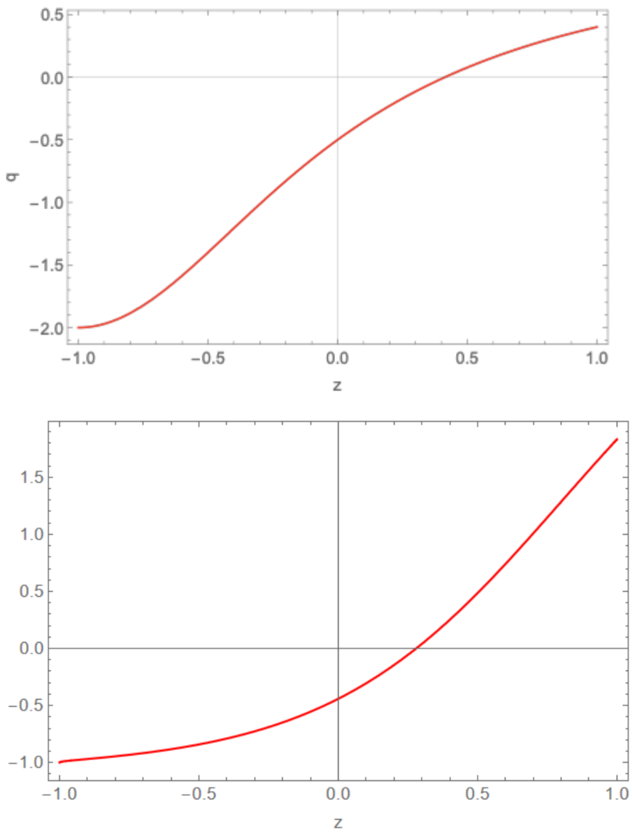



**Fig. 1** Behaviour of the deceleration parameter for $H = \frac{1}{t(2\lambda - t)(4\lambda - t)}$ (Upper Panel), $H = \nu + \frac{\mu}{t}$ (Lower Panel)

### 3.3 Jerk Parameter (j)

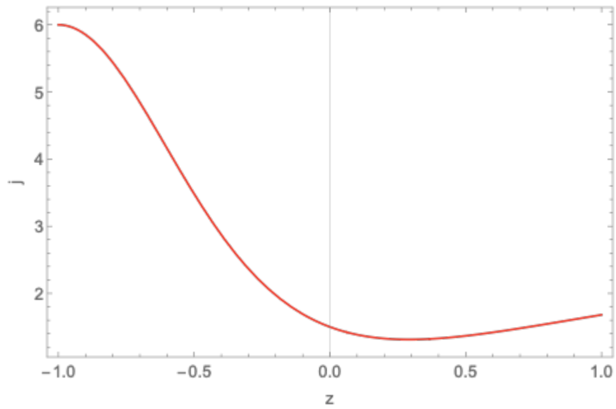
To obtain $j$ we use the relation $j = \frac{1}{a}\frac{d^3 a}{dt^3}H^{-3}$. For Eq. (10) we obtain $j$ as

$$j = 1 + 128\lambda^4 - 24\lambda^2 + 3\left(88\lambda^2 - 3\right)t^2 - 96\lambda t^3 + 12t^4 + 36\lambda\left(1 - 8\lambda^2\right)t, \quad (18)$$

and for Eq. (11) we obtain $q$ as

$$j = \frac{2\mu + (\mu + \nu t)\left[(\mu + \nu t)^2 - 3\mu\right]}{(\mu + \nu t)^3}. \quad (19)$$

We use Eqs. (18) and (19) to plot relation between $j$ and redshift parameter. The jerk parameter for Eq. (10) decreases slightly in the initial cosmic time and then increases in the later cosmic time (Upper Panel of Fig. 2) while for Eq. (11) it decreases in the initial cosmic time then increases slightly in the later cosmic time (Lower Panel of Fig. 2).



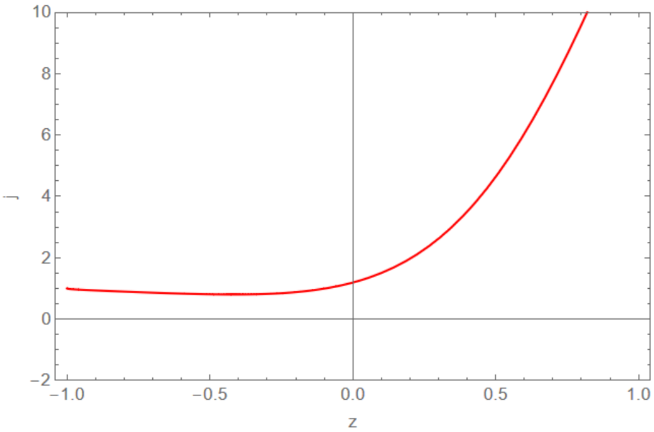




**Fig. 2** Behaviour of the Jerk parameter for $H = \frac{1}{t(2\lambda-t)(4\lambda-t)}$ (Upper Panel), $H = \nu + \frac{\mu}{t}$ (Lower Panel)

### 3.4 Snap Parameter(s)

To obtain $s$ we use the relation $s = \frac{1}{a}\frac{d^4a}{dt^4}H^{-4}$. For Eq. (10) we obtain $s$ as

$$s = (-3480\lambda^2 + 75)t^4 + (-600\lambda + 8640\lambda^3)t^3 + (-11904\lambda^4 + 1632\lambda^2 - 18)t^2 \\ +(9216\lambda^5 - 1728\lambda^3 + 72\lambda)t - 3072\lambda^6 + 704\lambda^4 - 48\lambda^2 + 1 + 720\lambda t^5 - 60t^6, \tag{20}$$

and for Eq. (11) we obtain $s$ as

$$s = \frac{(\mu-3)(\mu-2)(\mu-1)\mu + 4\nu^3\mu t^3 + 6\nu^2(\mu-1)\mu t^2 + \nu^4 t^4 + 4\nu(\mu-2)(\mu-1)\mu t}{(\mu+\nu t)^4}. \tag{21}$$

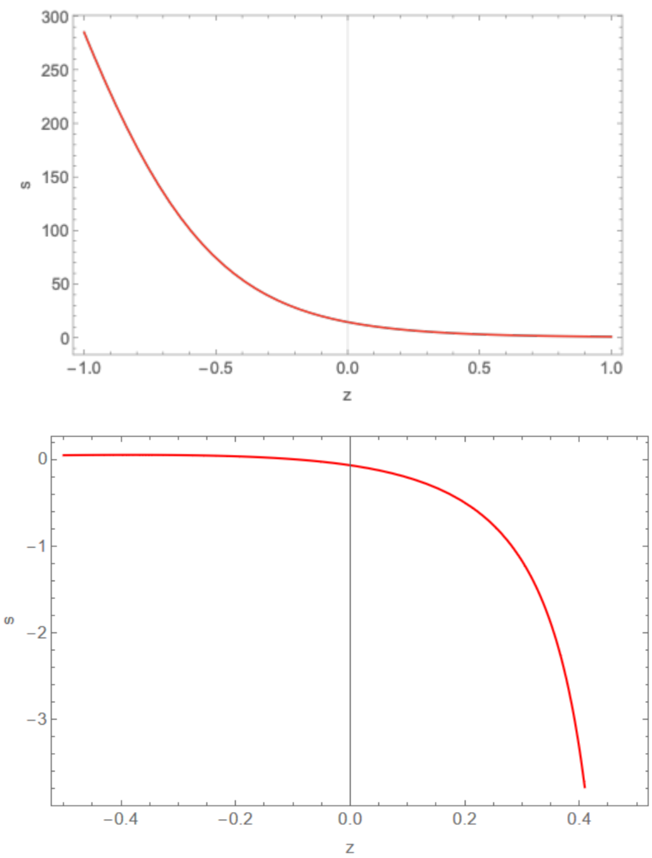



**Fig. 3** Behaviour of the Snap parameter for $H = \frac{1}{t(2\lambda-t)(4\lambda-t)}$ (Upper Panel), $H = \nu + \frac{\mu}{t}$ (Lower Panel)

We use Eqs. (20) and (21) to plot relation between snap parameter and redshift.

In the literature, there are two significant geometrical diagnostic approaches. They are the $Om(z)$ diagnostics and the determination of the state finder pair $(j, s)$ in the $j - s$ plane. These geometrical diagnostic techniques [25, 26] can help discriminate between dark energy theories. The snap parameter for Eq. (10) increases while the snap parameter for Eq. (11) decreases parabolically over the cosmic time.

### 3.5 Energy density ($\rho$)

We are using Eq. (7) to obtain $\rho$ from the respective Hubble parameters.

For Eq. (10) we obtain $\rho$ as

$$\rho = \frac{3}{t^2\left(8\lambda^2+t^2-6\lambda t\right)^2} - \frac{\alpha 6^n(2n-1)}{2}\left(\frac{1}{t^2\left(8\lambda^2+t^2-6\lambda t\right)^2}\right)^n. \tag{22}$$

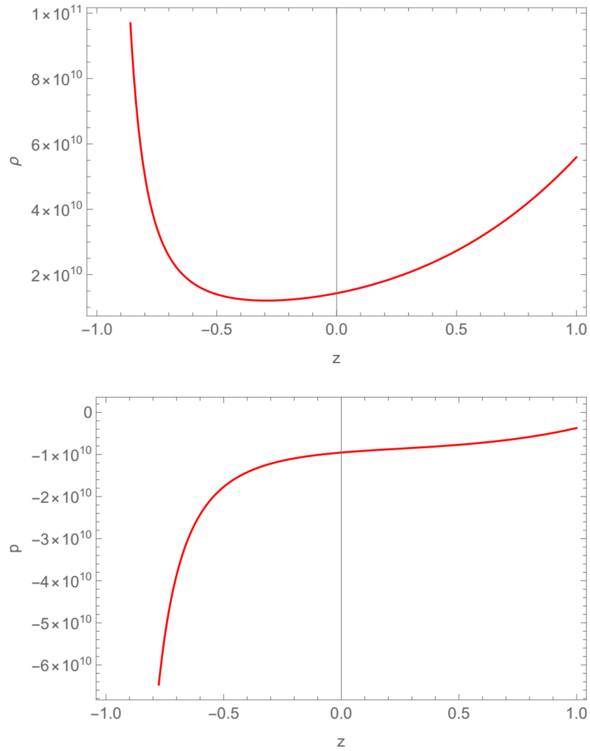



**Fig. 4** Behaviour of the energy density for $H = \frac{1}{t(2\lambda-t)(4\lambda-t)}$ (Upper Panel), $H = \nu + \frac{\mu}{t}$ (Lower Panel)

For Eq. (11) we obtain $\rho$ as

$$\rho = 3\left(\nu + \frac{\mu}{t}\right)^2 - \frac{\alpha 6^n(2n-1)}{2}\left(\nu + \frac{\mu}{t}\right)^{2n}. \tag{23}$$

We use Eqs. (22) and (23) to plot relation between energy density and redshift parameter.

The choice of $n$ in both the Eqs. (22) and (23) have been considered in such a manner that, the energy density remains positive throughout the cosmic evolution of the Universe.

3.6 Pressure ($p$)

We are using Eq. (8) to obtain $p$ from the respective Hubble parameters.

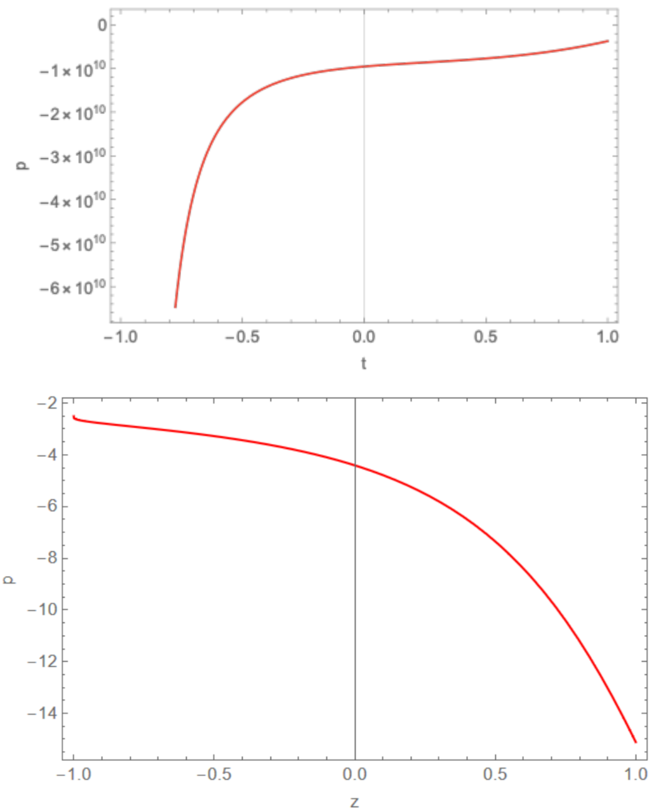



**Fig. 5** Behaviour of the pressure for $H = \frac{1}{t(2\lambda-t)(4\lambda-t)}$ (Upper Panel), $H = \nu + \frac{\mu}{t}$ (Lower Panel)

For Eq. (10) we obtain $p$ as

$$p = \frac{16\lambda^2 + 6t^2 - 24\lambda t - 3}{t^2\left(8\lambda^2 + t^2 - 6\lambda t\right)^2} - \alpha 6^{n-1}(2n-1)\left(\frac{1}{t^2\left(8\lambda^2 + t^2 - 6\lambda t\right)^2}\right)^n \left(2n\left(8\lambda^2 + 3t^2 - 12\lambda t\right) - 3\right). \tag{24}$$

For Eq. (11) we obtain $p$ as

$$p = \frac{4\mu + \alpha 2^n 3^{n-1}(1-2n)\left(2\mu n - 3(\mu+\nu t)^2\right)\left(\nu + \frac{\mu}{t}\right)^{2(n-1)} - 6(\mu+\nu t)^2}{2t^2}. \tag{25}$$

We use Eqs. (24) and (25) to plot relation between pressure and redshift parameter. The pressure is negative in both implying the force of anti gravity which explains the accelerated expansion of Universe.

### 3.7 Equation of State Parameter ($\omega$)

We are using Eq. (9) to obtain $\omega$ from the respective Hubble parameters.

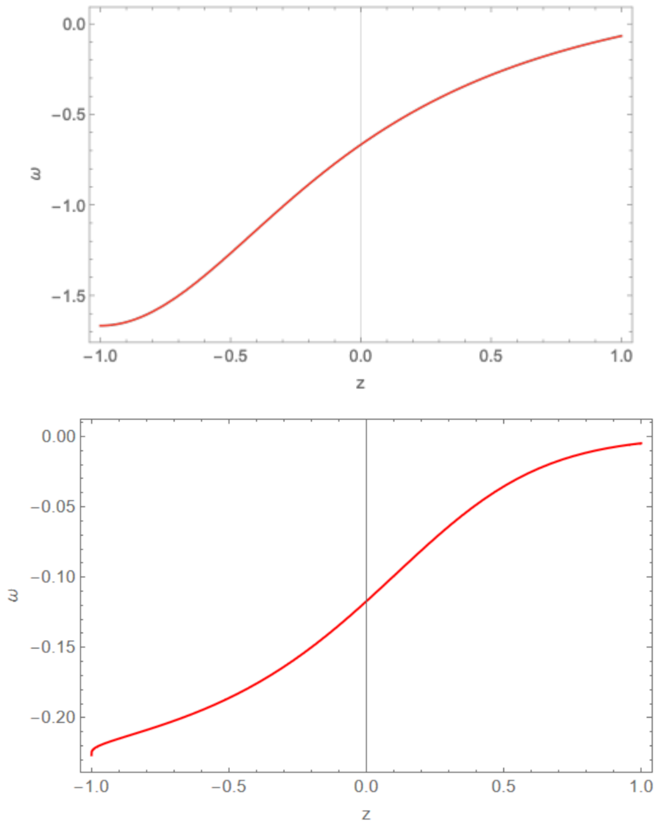



**Fig. 6** Behaviour of the EOS parameter for $H = \frac{1}{t(2\lambda-t)(4\lambda-t)}$ (Upper Panel), $H = \nu + \frac{\mu}{t}$ (Lower Panel)

For Eq. (10) we obtain $\omega$ as

$$\omega = \frac{\frac{6\left(16\lambda^2+6t^2-24\lambda t-3\right)}{t^2(8\lambda^2+t^2-6\lambda t)^2} - \alpha 6^n(2n-1)\left(\frac{1}{t^2(8\lambda^2+t^2-6\lambda t)^2}\right)^n\left(2n\left(8\lambda^2+3t^2-12\lambda t\right)-3\right)}{\frac{2}{t^2(8\lambda^2+t^2-6\lambda t)^2} - \frac{\alpha 6^n(2n-1)}{3}\left(\frac{1}{t^2(8\lambda^2+t^2-6\lambda t)^2}\right)^n}. \tag{26}$$

For Eq. (11) we obtain the EOS parameter ($\omega$) as

$$\omega = \frac{4\mu + \alpha 2^n 3^{n-1}(1-2n)\left(2\mu n - 3(\mu+\nu t)^2\right)\left(\nu+\frac{\mu}{t}\right)^{2(n-1)} - 6(\mu+\nu t)^2}{t^2\left(6\left(\nu+\frac{\mu}{t}\right)^2 - \alpha 6^n(2n-1)\left(\nu+\frac{\mu}{t}\right)^{2n}\right)}. \tag{27}$$

We use Eqs. (26) and (27) to plot relation between $\omega$ and redshift parameter. The equation of state parameter decreases from a value less than $\frac{-1}{3}$ at an initial epoch to negative values at late times in both the cases which follows in line with scientific observations. The parameters v and regulated the evolutionary behavior of the dynamical and EOS parameters. The first step was to constrain the scale factor parameters to get the geometrical parameters in the required range. Then the model parameter was constrained to produce the positive energy density (see Fig. 4 and Fig. 6).

### 3.8 Energy Conditions

Three energy conditions were checked to confirm the viability of both Hubble Parameters. Hence, we present here

(a) Null Energy Condition (NEC): $\rho + p \geq 0$,
(b) Weak Energy Condition (WEC): $\rho \geq 0$; $\rho + p \geq 0$,
(c) Strong Energy Condition (SEC): $\rho + 3p \geq 0$
(d) Dominant Energy Condition (DEC): $\rho - p \geq 0$.

The violation of the strong energy requirement has become so crucial in modified gravity theories, it is now threatened with extinction. The energy conditions NEC, WEC, SEC, and DEC for this $f(T)$ gravity model are currently as follows:

Except for the DEC, all models are expected to violate the energy conditions since they evolve in the phantom phase. Figures 7 and 8 graphically depicts the behavior of the energy conditions for the introduction of the Hubble parameters model in order. Only DEC is satisfied in the appropriate range for all three models, but WEC and SEC are violated as predicted. For Eq.(10) energy conditions are satisfied only in the later cosmic time and not during the initial epoch which does not support the observations where as for Eq. (11) all the energy conditions are satisfied during the entire cosmic time.

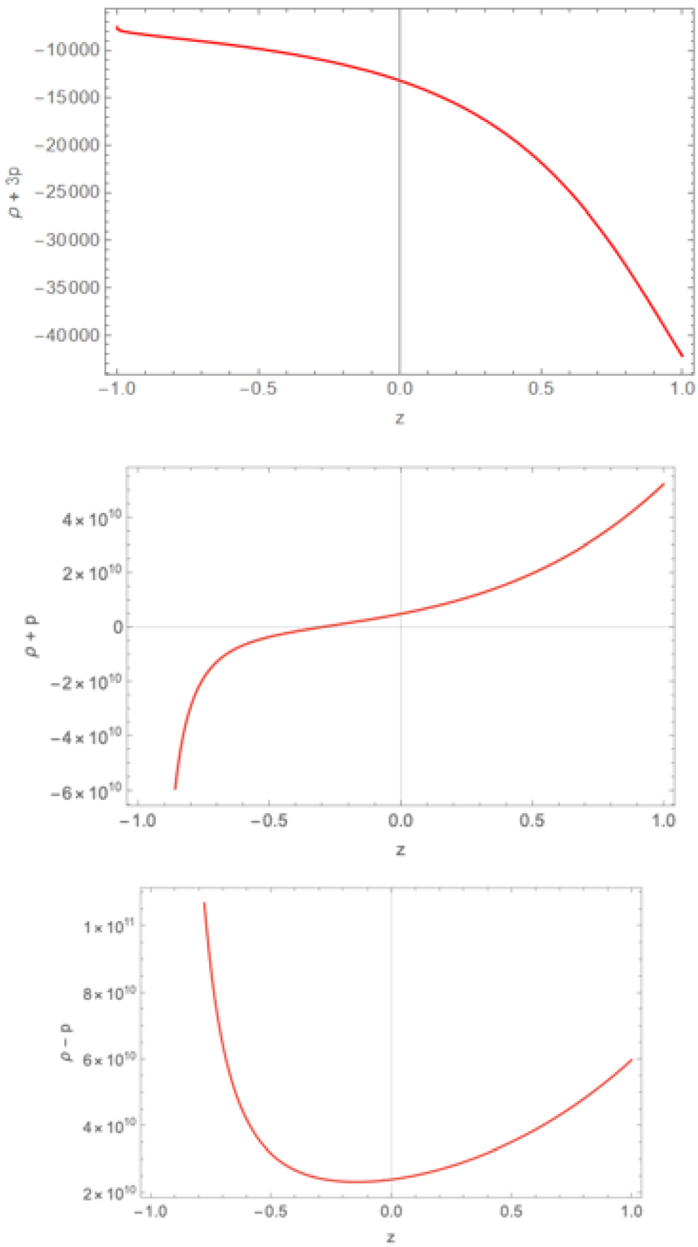



**Fig. 7** Behaviour of the Energy Conditions for $H = \frac{1}{t(2\lambda - t)(4\lambda - t)}$

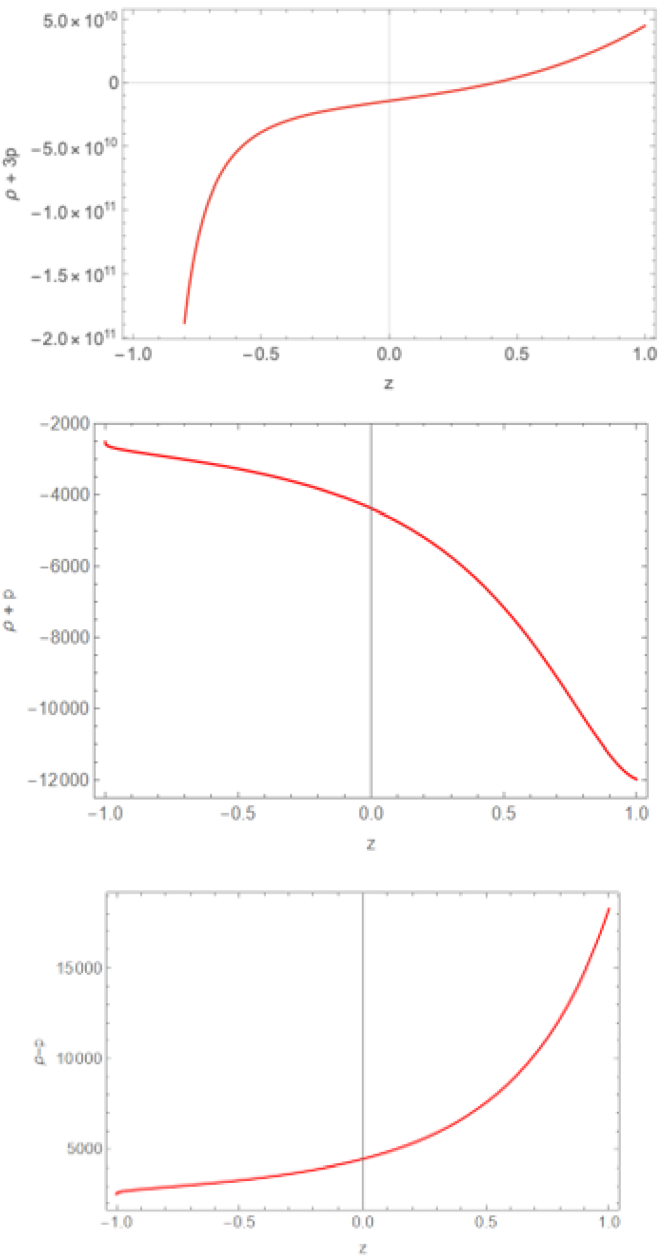



**Fig. 8** Behaviour of the Energy Conditions for $H = \nu + \frac{\mu}{t}$

## 4 Conclusion

The physical parameters of the cosmological models are derived using the Hubble parameter $H(t) = \frac{1}{t(2\lambda-t)(4\lambda-t)}$ and $H(t) = \nu + \frac{\mu}{t}$. The equation of state parameter, from where the nature of Universe during evolution would be known, has been derived with respect to the cosmic time. The energy conditions for these two scale factors are derived along with the physical parameters such as deceleration parameter, snap parameter and jerk parameter are also derived with respect to the cosmic time. The scale factor used in this yields a deceleration parameter that is positive early and negative at late time. Relation between cosmic time $t$ and redshift parameter $z$ is also been derived. The graphical representation of the parameters were presented and their behaviours were analyzed. The accelerating behavior of the models under a modified theory of gravity is further validated by the violation of SEC.

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